

SOLUTIONS TO 18.01 EXERCISES

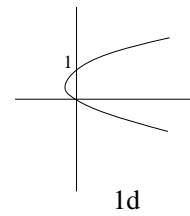
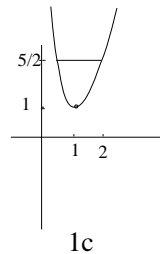
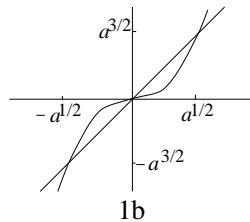
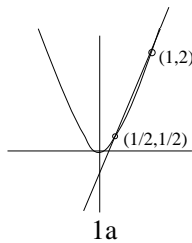
Unit 4. Applications of integration

4A. Areas between curves.

4A-1 a) $\int_{1/2}^1 (3x - 1 - 2x^2)dx = (3/2)x^2 - x - (2/3)x^3 \Big|_{1/2}^1 = 1/24$

b) $x^3 = ax \implies x = \pm a$ or $x = 0$. There are two enclosed pieces ($-a < x < 0$ and $0 < x < a$) with the same area by symmetry. Thus the total area is:

$$2 \int_0^{\sqrt{a}} (ax - x^3)dx = ax^2 - (1/2)x^4 \Big|_0^{\sqrt{a}} = a^2/2$$



c) $x + 1/x = 5/2 \implies x^2 + 1 = 5x/2 \implies x = 2$ or $1/2$. Therefore, the area is

$$\int_{1/2}^2 [5/2 - (x + 1/x)]dx = 5x/2 - x^2/2 - \ln x \Big|_{1/2}^2 = 15/8 - 2 \ln 2$$

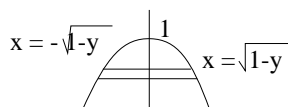
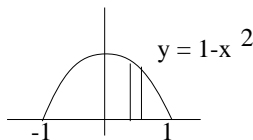
d) $\int_0^1 (y - y^2)dy = y^2/2 - y^3/3 \Big|_0^1 = 1/6$

4A-2 First way (dx):

$$\int_{-1}^1 (1 - x^2)dx = 2 \int_0^1 (1 - x^2)dx = 2x - 2x^3/3 \Big|_0^1 = 4/3$$

Second way (dy): ($x = \pm\sqrt{1-y}$)

$$\int_0^1 2\sqrt{1-y}dy = (4/3)(1-y)^{3/2} \Big|_0^1 = 4/3$$

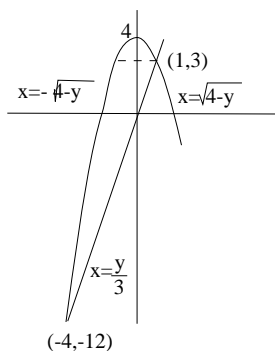


4A-3 $4 - x^2 = 3x \implies x = 1$ or -4 . First way (dx):

$$\int_{-4}^1 (4 - x^2 - 3x) dx = 4x - x^3/3 - 3x^2/2 \Big|_{-4}^1 = 125/6$$

Second way (dy): Lower section has area

$$\int_{-12}^3 (y/3 + \sqrt{4-y}) dy = y^2/6 - (2/3)(4-y)^{3/2} \Big|_{-12}^3 = 117/6$$



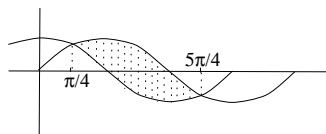
Upper section has area

$$\int_3^4 2\sqrt{4-y} dy = -(4/3)(4-y)^{3/2} \Big|_3^4 = 4/3$$

(See picture for limits of integration.) Note that $117/6 + 4/3 = 125/6$.

4A-4 $\sin x = \cos x \implies x = \pi/4 + k\pi$. So the area is

$$\int_{\pi/4}^{5\pi/4} (\sin x - \cos x) dx = (-\cos x - \sin x) \Big|_{\pi/4}^{5\pi/4} = 2\sqrt{2}$$



4B. Volumes by slicing; volumes of revolution

$$\begin{aligned}
 \text{4B-1 a) } \int_{-1}^1 \pi y^2 dx &= \int_{-1}^1 \pi(1-x^2)^2 dx = 2\pi \int_0^1 (1-2x^2+x^4) dx \\
 &= 2\pi(x - 2x^3/3 + x^5/5) \Big|_0^1 = 16\pi/15
 \end{aligned}$$

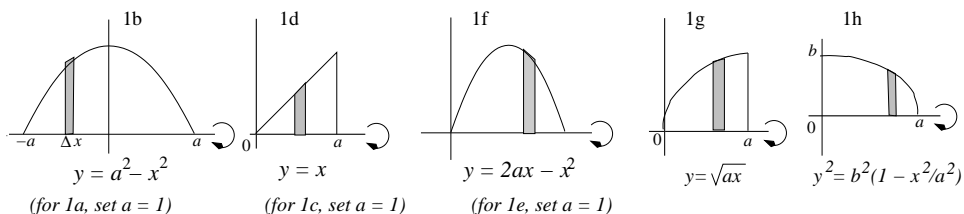
$$\begin{aligned}
 \text{b) } \int_{-a}^a \pi y^2 dx &= \int_{-a}^a \pi(a^2-x^2)^2 dx = 2\pi \int_0^a (a^4 - 2a^2x^2 + x^4) dx \\
 &= 2\pi(a^4x - 2a^2x^3/3 + x^5/5) \Big|_0^a = 16\pi a^5/15
 \end{aligned}$$

$$\text{c) } \int_0^1 \pi x^2 dx = \pi/3$$

$$\text{d) } \int_0^a \pi x^2 dx = \pi a^3/3$$

$$\begin{aligned}
 \text{e) } \int_0^2 \pi(2x-x^2)^2 dx &= \int_0^2 \pi(4x^2-4x^3+x^4) dx = \pi(4x^3/3 - x^4 + x^5/5) \Big|_0^2 = \\
 &16\pi/15
 \end{aligned}$$

(Why (e) the same as (a)? Complete the square and translate.)



$$\text{f) } \int_0^{2a} \pi(2ax - x^2)^2 dx = \int_0^{2a} \pi(4a^2x^2 - 4ax^3 + x^4) dx$$

$$= \pi(4a^2x^3/3 - ax^4 + x^5/5)|_0^2 = 16\pi a^5/15$$

(Why is (f) the same as (b)? Complete the square and translate.)

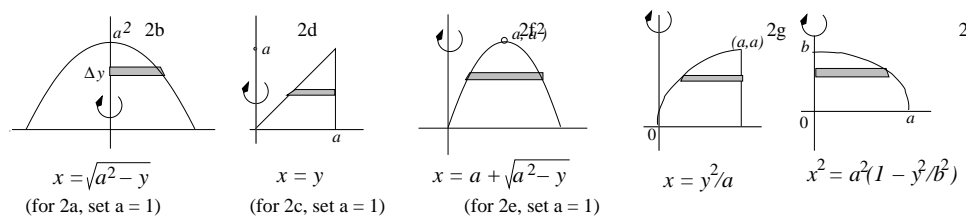
$$\text{g) } \int_0^a ax dx = \pi a^3/2$$

$$\text{h) } \int_0^a \pi y^2 dx = \int_0^a \pi b^2(1 - x^2/a^2) dx = \pi b^2(x - x^3/3a^2)|_0^a = 2\pi b^2 a/3$$

$$\begin{aligned} \text{4B-2 a) } \int_0^1 \pi(1-y) dy &= \pi/2 & \text{b) } \int_0^{a^2} \pi(a^2-y) dy &= \pi a^4/2 \\ \text{c) } \int_0^1 \pi(1-y^2) dy &= 2\pi/3 & \text{d) } \int_0^a \pi(a^2-y^2) dy &= 2\pi a^3/3 \\ \text{e) } x^2 - 2x + y = 0 &\implies x = 1 \pm \sqrt{1-y}. \end{aligned}$$

$$\begin{aligned} \int_0^1 \pi[(1 + \sqrt{1-y})^2 - (1 - \sqrt{1-y})^2] dy &= \int_0^1 4\pi\sqrt{1-y} dy \\ &= -(8/3)\pi(1-y)^{3/2} \Big|_0^1 = 8\pi/3 \end{aligned}$$

(In contrast with 1(e) and 1(a), rotation around the y -axis makes the solid in 2(e) different from 2(a).)



$$\text{f) } x^2 - 2ax + y = 0 \implies x = a \pm \sqrt{a^2 - y}. \text{ Using the method of washers:}$$

$$\begin{aligned} \int_0^{a^2} \pi[(a + \sqrt{a^2 - y})^2 - (a - \sqrt{a^2 - y})^2] dy &= \int_0^{a^2} 4\pi a \sqrt{a^2 - y} dy \\ &= -(8/3)\pi a(a^2 - y)^{3/2} \Big|_0^{a^2} = 8\pi a^4/3 \end{aligned}$$

g) Using washers:

$$\int_0^a \pi(a^2 - (y^2/a)^2) dy = \pi(a^2 y - y^5/5a^2) \Big|_0^a = 4\pi a^3/5.$$

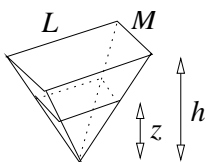
h)

$$\int_{-b}^b \pi x^2 dy = 2\pi \int_0^b a^2(1 - y^2/b^2) dy = 2\pi(a^2 y - a^2 y^3/3b^2) \Big|_0^b = 4\pi a^2 b/3$$

(The answer in 2(h) is double the answer in 1(h), with a and b reversed. Can you see why?)

4B-3 Put the pyramid upside-down. By similar triangles, the base of the smaller bottom pyramid has sides of length $(z/h)L$ and $(z/h)M$.

The base of the big pyramid has area $b = LM$; the base of the smaller pyramid forms a cross-sectional slice, and has area

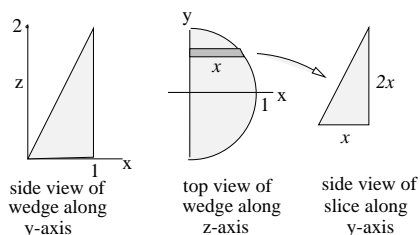


$$(z/h)L \cdot (z/h)M = (z/h)^2 LM = (z/h)^2 b$$

Therefore, the volume is

$$\int_0^h (z/h)^2 b dz = bz^3/3h^2 \Big|_0^h = bh/3$$

4B-4 The slice perpendicular to the xz -plane are right triangles with base of length x and height $z = 2x$. Therefore the area of a slice is x^2 . The volume is

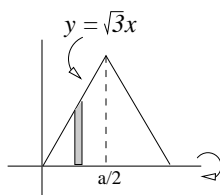


$$\int_{-1}^1 x^2 dy = \int_{-1}^1 (1 - y^2) dy = 4/3$$

4B-5 One side can be described by $y = \sqrt{3}x$ for $0 \leq x \leq a/2$.

Therefore, the volume is

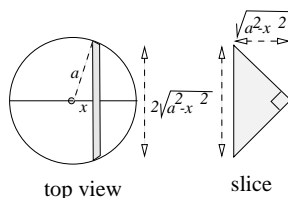
$$2 \int_0^{a/2} \pi y^2 dx = 2 \int_0^{a/2} \pi (\sqrt{3}x)^2 dx = \pi a^3/4$$



4B-6 If the hypotenuse of an isosceles right triangle has length h , then its area is $h^2/4$. The endpoints of the slice in the xy -plane are $y = \pm\sqrt{a^2 - x^2}$, so $h = 2\sqrt{a^2 - x^2}$. In all the volume is

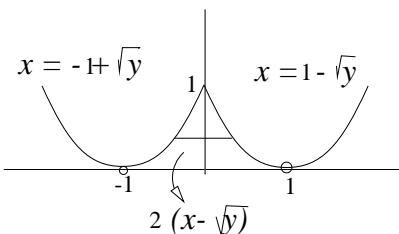
$$\int_{-a}^a (h^2/4)dx = \int_{-a}^a (a^2 - x^2)dx = 4a^3/3$$

4B-7 Solving for x in $y = (x - 1)^2$ and $y = (x + 1)^2$ gives the values



$$x = 1 \pm \sqrt{y} \quad \text{and} \quad x = -1 \pm \sqrt{y}$$

The hard part is deciding which sign of the square root representing the endpoints of the square.



Method 1: The point $(0, 1)$ has to be on the two curves. Plug in $y = 1$ and $x = 0$ to see that the square root must have the opposite sign from 1: $x = 1 - \sqrt{y}$ and $x = -1 + \sqrt{y}$.

Method 2: Look at the picture. $x = 1 + \sqrt{y}$ is the wrong choice because it is the right half of the parabola with vertex $(1, 0)$. We want the left half: $x = 1 - \sqrt{y}$. Similarly, we want $x = -1 + \sqrt{y}$, the right half of the parabola with vertex $(-1, 0)$. Hence, the side of the square is the interval $-1 + \sqrt{y} \leq x \leq 1 - \sqrt{y}$, whose length is $2(1 - \sqrt{y})$, and the

$$\text{Volume} = \int_0^1 (2(1 - \sqrt{y}))^2 dy = 4 \int_0^1 (1 - 2\sqrt{y} + y) dy = 2/3 .$$

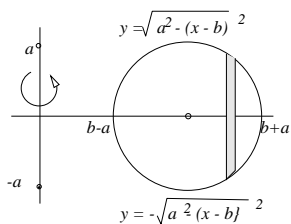
4C. Volumes by shells

4C-1 a)

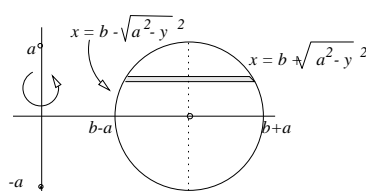
$$\text{Shells: } \int_{b-a}^{b+a} (2\pi x)(2y)dx = \int_{b-a}^{b+a} 4\pi x \sqrt{a^2 - (x-b)^2} dx$$

$$\text{b) } (x-b)^2 = a^2 - y^2 \implies x = b \pm \sqrt{a^2 - y^2}$$

$$\begin{aligned} \text{Washers: } \int_{-a}^a \pi(x_2^2 - x_1^2)dy &= \int_{-a}^a \pi((b + \sqrt{a^2 - y^2})^2 - (b - \sqrt{a^2 - y^2})^2)dy \\ &= \pi \int_{-a}^a 4b\sqrt{a^2 - y^2} dy \end{aligned}$$



Shells



Washers

$$\text{c) } \int_{-a}^a \sqrt{a^2 - y^2} dy = \pi a^2 / 2, \text{ because it's the area of a semicircle of radius } a.$$

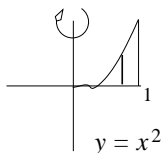
$$\text{Thus (b) } \implies \text{Volume of torus} = 2\pi^2 a^2 b$$

$$\text{d) } z = x - b, dz = dx$$

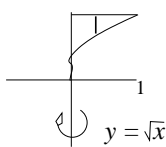
$$\int_{b-a}^{b+a} 4\pi x \sqrt{a^2 - (x-b)^2} dx = \int_{-a}^a 4\pi(z+b) \sqrt{a^2 - z^2} dz = \int_{-a}^a 4\pi b \sqrt{a^2 - z^2} dz$$

because the part of the integrand with the factor z is odd, and so it integrates to 0.

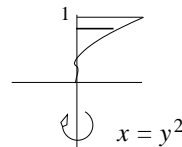
$$\text{4C-2 } \int_0^1 2\pi xy dx = \int_0^1 2\pi x^3 dx = \pi/2$$



4C-2 (shells)



4C-3a (shells)



4C-3b (discs)

4C-3 Shells:
$$\int_0^1 2\pi x(1-y)dx = \int_0^1 2\pi x(1-\sqrt{x})dx = \pi/5$$

Disks:
$$\int_0^1 \pi x^2 dy = \int_0^1 \pi y^4 dy = \pi/5$$

4C-4 a)
$$\int_0^1 2\pi y(2x)dy = 4\pi \int_0^1 y\sqrt{1-y}dy$$

b)
$$\int_0^{a^2} 2\pi y(2x)dy = 4\pi \int_0^{a^2} y\sqrt{a^2-y}dy$$

c)
$$\int_0^1 2\pi y(1-y)dy$$

d)
$$\int_0^a 2\pi y(a-y)dy$$

e) $x^2 - 2x + y = 0 \implies x = 1 \pm \sqrt{1-y}$.

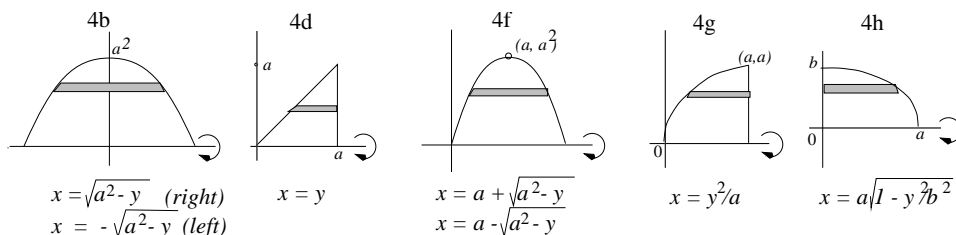
The interval $1 - \sqrt{1-y} \leq x \leq 1 + \sqrt{1-y}$ has length $2\sqrt{1-y}$

$$\implies V = \int_0^1 2\pi y(2\sqrt{1-y})dy = 4\pi \int_0^1 y\sqrt{1-y}dy$$

f) $x^2 - 2ax + y = 0 \implies x = a \pm \sqrt{a^2-y}$.

The interval $a - \sqrt{a^2-y} \leq x \leq a + \sqrt{a^2-y}$ has length $2\sqrt{a^2-y}$

$$\implies V = \int_0^{a^2} 2\pi y(2\sqrt{a^2-y})dy = 4\pi \int_0^{a^2} y\sqrt{a^2-y}dy$$



g)
$$\int_0^a 2\pi y(a - y^2/a)dy$$

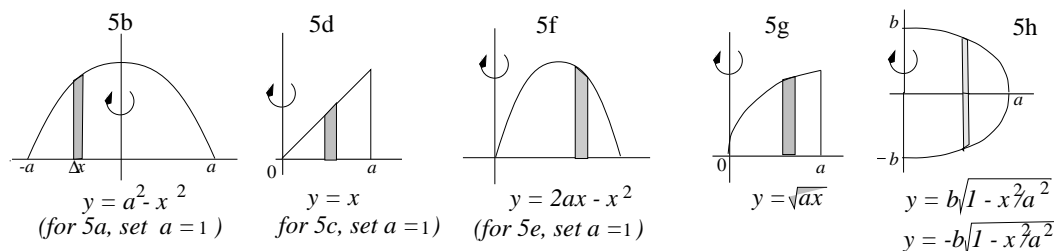
$$\text{h) } \int_0^b 2\pi xy dy = \int_0^b 2\pi y(a^2(1 - y^2/b^2)) dy$$

(Why is the lower limit of integration 0 rather than $-b$?)

$$\mathbf{4C-5} \quad \text{a) } \int_0^1 2\pi x(1 - x^2) dx \quad \text{c) } \int_0^1 2\pi xy dx = \int_0^1 2\pi x^2 dx$$

$$\text{b) } \int_0^a 2\pi x(a^2 - x^2) dx \quad \text{d) } \int_0^a 2\pi xy dx = \int_0^a 2\pi x^2 dx$$

$$\text{e) } \int_0^2 2\pi xy dx = \int_0^2 2\pi x(2x - x^2) dx \quad \text{f) } \int_0^{2a} 2\pi xy dx = \int_0^{2a} 2\pi x(ax -$$



$$x^2) dx \quad \text{g) } \int_0^a 2\pi xy dx = \int_0^a 2\pi x\sqrt{ax} dx$$

$$\text{h) } \int_0^a 2\pi x(2y) dx = \int_0^a 2\pi x(2b^2(1 - x^2/a^2)) dx$$

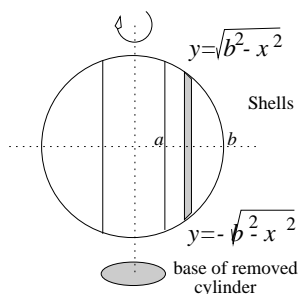
(Why did y get doubled this time?)

4C-6

$$\begin{aligned} \int_a^b 2\pi x(2y) dx &= \int_a^b 2\pi x(2\sqrt{b^2 - x^2}) dx \\ &= -(4/3)\pi(b^2 - x^2)^{3/2} \Big|_a^b = (4\pi/3)(b^2 - a^2)^{3/2} \end{aligned}$$

4D. Average value

4D-1 Cross-sectional area at x is $= \pi y^2 = \pi \cdot (x^2)^2 = \pi x^4$. Therefore,



$$\text{average cross-sectional area} = \frac{1}{2} \int_0^2 \pi x^4 dx = \frac{\pi x^5}{10} \Big|_0^2 = \frac{16\pi}{5}.$$

$$\mathbf{4D-2} \quad \text{Average} = \frac{1}{a} \int_a^{2a} \frac{dx}{x} = \frac{1}{a} \ln x \Big|_a^{2a} = \frac{1}{a} (\ln 2a - \ln a) = \frac{1}{a} \ln \left(\frac{2a}{a} \right) = \frac{\ln 2}{a}.$$

4D-3 Let $s(t)$ be the distance function; then the velocity is $v(t) = s'(t)$

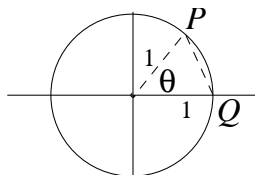
$$\begin{aligned} \text{Average value of velocity} &= \frac{1}{b-a} \int_a^b s'(t) dt = \frac{s(b) - s(a)}{b-a} \text{ by FT1} \\ &= \text{average velocity over time interval } [a, b] \end{aligned}$$

4D-4 By symmetry, we can restrict P to the upper semicircle.

By the law of cosines, we have $|PQ|^2 = 1^2 + 1^2 - 2 \cos \theta$. Thus

$$\text{average of } |PQ|^2 = \frac{1}{\pi} \int_0^\pi (2 - 2 \cos \theta) d\theta = \frac{1}{\pi} [2\theta - 2 \sin \theta]_0^\pi = 2$$

(This is the value of $|PQ|^2$ when $\theta = \pi/2$, so the answer is reasonable.)



4D-5 By hypothesis, $g(x) = \frac{1}{x} \int_0^x f(t) dt$. To express $f(x)$ in terms of $g(x)$, multiply through by x and apply the Sec. Fund. Thm:

$$\int_0^x f(t)dt = xg(x) \Rightarrow f(x) = g(x) + xg'(x), \text{ by FT2.}$$

$$\mathbf{4D-6} \text{ Average value of } A(t) = \frac{1}{T} \int_0^T A_0 e^{rt} dt = \frac{1}{T} \frac{A_0}{r} e^{rt} \Big|_0^T = \frac{A_0}{rT} (e^{rT} - 1)$$

If rT is small, we can approximate: $e^{rT} \approx 1 + rT + \frac{(rT)^2}{2}$, so we get

$$A(t) \approx \frac{A_0}{rT} \left(rT + \frac{(rT)^2}{2} \right) = A_0 \left(1 + \frac{rT}{2} \right).$$

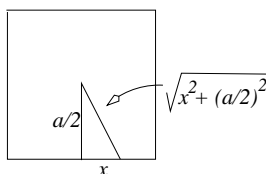
(If $T \approx 0$, at the end of T years the interest added will be $A_0 rT$; thus the average is approximately what the account grows to in $T/2$ years, which seems reasonable.)

$$\mathbf{4D-7} \quad \frac{1}{b} \int_0^b x^2 dx = b^2/3$$

4D-8 The average on each side is the same as the average

over all four sides. Thus the average distance is

$$\frac{1}{a} \int_{-a/2}^{a/2} \sqrt{x^2 + (a/2)^2} dx$$



Can't be evaluated by a formula until Unit 5. The average of the square of the distance is

$$\frac{1}{a} \int_{-a/2}^{a/2} (x^2 + (a/2)^2) dx = \frac{2}{a} \int_0^{a/2} (x^2 + (a/2)^2) dx = a^2/3$$

$$\mathbf{4D-9} \quad \frac{1}{\pi/a} \int_0^{\pi/a} \sin ax dx - \frac{1}{\pi} \cos(ax) \Big|_0^{\pi/a} = 2/\pi$$

4D'. Work

4D'-1 According to Hooke's law, we have $F = kx$, where F is the force, x is the displacement (i.e., the added length), and k is the Hooke's law constant for the

spring.

To find k , substitute into Hooke's law: $2,000 = k \cdot (1/2) \Rightarrow k = 4000$.

To find the work W , we have

$$W = \int_0^6 F dx = \int_0^6 4000x dx = 2000x^2 \Big|_0^6 = 72,000 \text{ inch-pounds} = 6,000 \text{ foot-pounds.}$$

4D'-2 Let $W(h)$ = weight of pail and paint at height h .

$W(0) = 12$, $W(30) = 10 \Rightarrow W(h) = 12 - \frac{1}{15}h$, since the pulling and leakage both occur at a constant rate.

$$\text{work} = \int_0^{30} W(h) dh = \int_0^{30} \left(12 - \frac{h}{15}\right) dh = 12h - \frac{h^2}{30} \Big|_0^{30} = 330 \text{ ft-lbs.}$$

4D'-3 Think of the hose as divided into many equal little infinitesimal pieces, of length dh , each of which must be hauled up to the top of the building.

The piece at distance h from the top end has weight $2 dh$; to haul it up to the top requires $2h dh$ ft-lbs. Adding these up,

$$\text{total work} = \int_0^{50} 2h dh = h^2 \Big|_0^{50} = 2500 \text{ ft-lbs.}$$

4D'-4 If they are x units apart, the gravitational force between them is $\frac{g m_1 m_2}{x^2}$.

$$\text{work} = \int_d^{nd} \frac{g m_1 m_2}{x^2} dx = -\frac{g m_1 m_2}{x} \Big|_d^{nd} = -g m_1 m_2 \left(\frac{1}{nd} - \frac{1}{d} \right) = \frac{g m_1 m_2}{d} \left(\frac{n-1}{n} \right).$$

The limit as $n \rightarrow \infty$ is $\frac{g m_1 m_2}{d}$.

4E. Parametric equations

4E-1 $y - x = t^2$, $y - 2x = -t$. Therefore,

$$y - x = (y - 2x)^2 \implies y^2 - 4xy + 4x^2 - y + x = 0 \quad (\text{parabola})$$

4E-2 $x^2 = t^2 + 2 + 1/t^2$ and $y^2 = t^2 - 2 + 1/t^2$. Subtract, getting the hyperbola $x^2 - y^2 = 4$

4E-3 $(x - 1)^2 + (y - 4)^2 = \sin^2 \theta + \cos^2 t = 1$ (circle)

4E-4 $1 + \tan^2 t = \sec^2 t \implies 1 + x^2 = y^2$ (hyperbola)

4E-5 $x = \sin 2t = 2 \sin t \cos t = \pm 2\sqrt{1 - y^2}y$. This gives $x^2 = 4y^2 - 4y^4$.

4E-6 $y' = 2x$, so $t = 2x$ and

$$x = t/2, \quad y = t^2/4$$

4E-7 Implicit differentiation gives $2x + 2yy' = 0$, so that $y' = -x/y$. So the parameter is $t = -x/y$. Substitute $x = -ty$ in $x^2 + y^2 = a^2$ to get

$$t^2y^2 + y^2 = a^2 \implies y^2 = a^2/(1+t^2)$$

Thus

$$y = \frac{a}{\sqrt{1+t^2}}, \quad x = \frac{-at}{\sqrt{1+t^2}}$$

For $-\infty < t < \infty$, this parametrization traverses the upper semicircle $y > 0$ (going clockwise). One can also get the lower semicircle (also clockwise) by taking the negative square root when solving for y ,

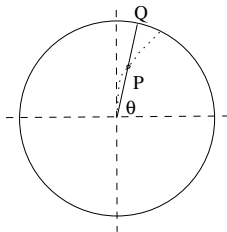
$$y = \frac{-a}{\sqrt{1+t^2}}, \quad x = \frac{at}{\sqrt{1+t^2}}$$

4E-8 The tip Q of the hour hand is given in terms of the angle θ by $Q = (\cos \theta, \sin \theta)$ (units are meters).

Next we express θ in terms of the time parameter t (hours). We have

$$\theta = \left\{ \begin{array}{l} \pi/2, t = 0 \\ \pi/3, t = 1 \end{array} \right\} \theta \text{ decreases linearly with } t$$

$$\implies \theta - \frac{\pi}{2} = \frac{\frac{\pi}{3} - \frac{\pi}{2} \cdot (t-0)}{1-0}. \text{ Thus we get } \theta = \frac{\pi}{2} - \frac{\pi}{6}t.$$



Finally, for the snail's position P , we have

$P = (t \cos \theta, t \sin \theta)$, where t increases from 0 to 1. So,

$$x = t \cos\left(\frac{\pi}{2} - \frac{\pi}{6}t\right) = t \sin \frac{\pi}{6}t, \quad y = t \sin\left(\frac{\pi}{2} - \frac{\pi}{6}t\right) = t \cos \frac{\pi}{6}t$$

4F. Arclength

4F-1 a) $ds = \sqrt{1+(y')^2}dx = \sqrt{26}dx$. Arclength $= \int_0^1 \sqrt{26}dx = \sqrt{26}$.

$$\text{b) } ds = \sqrt{1 + (y')^2} dx = \sqrt{1 + (9/4)x} dx.$$

$$\text{Arclength} = \int_0^1 \sqrt{1 + (9/4)x} dx = (8/27)(1 + 9x/4)^{3/2} \Big|_0^1 = (8/27)((13/4)^{3/2} - 1)$$

$$\text{c) } y' = -x^{-1/3}(1 - x^{2/3})^{1/2} = -\sqrt{x^{-2/3} - 1}. \text{ Therefore, } ds = x^{-1/3} dx, \text{ and}$$

$$\text{Arclength} = \int_0^1 x^{-1/3} dx = (3/2)x^{2/3} \Big|_0^1 = 3/2$$

$$\text{d) } y' = x(2 + x^2)^{1/2}. \text{ Therefore, } ds = \sqrt{1 + 2x^2 + x^4} dx = (1 + x^2) dx \text{ and}$$

$$\text{Arclength} = \int_1^2 (1 + x^2) dx = x + x^3/3 \Big|_1^2 = 10/3$$

4F-2 $y' = (e^x - e^{-x})/2$, so the hint says $1 + (y')^2 = y^2$ and $ds = \sqrt{1 + (y')^2} dx = y dx$. Thus,

$$\text{Arclength} = (1/2) \int_0^b (e^x + e^{-x}) dx = (1/2)(e^x - e^{-x}) \Big|_0^b = (e^b - e^{-b})/2$$

4F-3 $y' = 2x$, $\sqrt{1 + (y')^2} = \sqrt{1 + 4x^2}$. Hence, arclength = $\int_0^b \sqrt{1 + 4x^2} dx$.

4F-4 $ds = \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \sqrt{4t^2 + 9t^4} dt$. Therefore,

$$\begin{aligned} \text{Arclength} &= \int_0^2 \sqrt{4t^2 + 9t^4} dt = \int_0^2 (4 + 9t^2)^{1/2} t dt \\ &= (1/27)(4 + 9t^2)^{3/2} \Big|_0^2 = (40^{3/2} - 8)/27 \end{aligned}$$

4F-5 $dx/dt = 1 - 1/t^2$, $dy/dt = 1 + 1/t^2$. Thus

$$ds = \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \sqrt{2 + 2/t^4} dt \quad \text{and}$$

$$\text{Arclength} = \int_1^2 \sqrt{2 + 2/t^4} dt$$

4F-6 a) $dx/dt = 1 - \cos t$, $dy/dt = \sin t$.

$$ds/dt = \sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{2 - 2 \cos t} \quad (\text{speed of the point})$$

Forward motion (dx/dt) is largest for t an odd multiple of π ($\cos t = -1$). Forward motion is smallest for t an even multiple of π ($\cos t = 1$). (continued \rightarrow)

Remark: The largest forward motion is when the point is at the top of the wheel and the smallest is when the point is at the bottom (since $y = 1 - \cos t$).

$$\text{b) } \int_0^{2\pi} \sqrt{2 - 2 \cos t} dt = \int_0^{2\pi} 2 \sin(t/2) dt = -4 \cos(t/2) \Big|_0^{2\pi} = 8$$

$$\mathbf{4F-7} \quad \int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} dt$$

$$\mathbf{4F-8} \quad dx/dt = e^t(\cos t - \sin t), \quad dy/dt = e^t(\cos t + \sin t).$$

$$ds = \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2} dt = e^t \sqrt{2 \cos^2 t + 2 \sin^2 t} dt = \sqrt{2} e^t dt$$

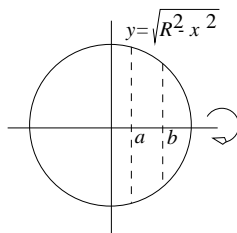
Therefore, the arclength is

$$\int_0^{10} \sqrt{2} e^t dt = \sqrt{2}(e^{10} - 1)$$

4G. Surface Area

4G-1 The curve $y = \sqrt{R^2 - x^2}$ for $a \leq x \leq b$ is revolved around the x -axis.

Since we have $y' = -x/\sqrt{R^2 - x^2}$, we get



$$ds = \sqrt{1 + (y')^2} dx = \sqrt{1 + x^2/(R^2 - x^2)} dx = \sqrt{R^2/(R^2 - x^2)} dx = (R/y) dx$$

Therefore, the area element is

$$dA = 2\pi y ds = 2\pi R dx$$

and the area is

$$\int_a^b 2\pi R dx = 2\pi R(b - a)$$

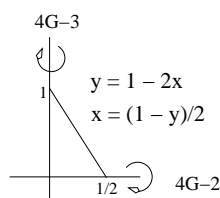
4G-2 Limits are $0 \leq x \leq 1/2$. $ds = \sqrt{5} dx$, so

$$dA = 2\pi y ds = 2\pi(1 - 2x)\sqrt{5} dx \implies A = 2\pi\sqrt{5} \int_0^{1/2} (1 - 2x) dx = \sqrt{5}\pi/2$$

4G-3 Limits are $0 \leq y \leq 1$. $x = (1 - y)/2$, $dx/dy = -1/2$. Thus

$$ds = \sqrt{1 + (dx/dy)^2} dy = \sqrt{5/4} dy;$$

$$dA = 2\pi y ds = \pi(1 - y)(\sqrt{5}/2) dy \implies A = (\sqrt{5}\pi/2) \int_0^1 (1 - y) dy = \sqrt{5}\pi/4$$



$$4\mathbf{G-4} \quad A = \int 2\pi y ds = \int_0^4 2\pi x^2 \sqrt{1 + 4x^2} dx$$

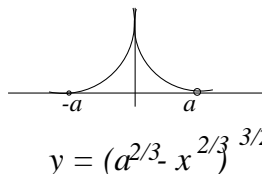
$$4\mathbf{G-5} \quad x = \sqrt{y}, \quad dx/dy = -1/2\sqrt{y}, \quad \text{and} \quad ds = \sqrt{1 + 1/4y} dy$$

$$\begin{aligned} A &= \int 2\pi x ds = \int_0^2 2\pi \sqrt{y} \sqrt{1 + 1/4y} dy \\ &= \int_0^2 2\pi \sqrt{y + 1/4} dy \\ &= (4\pi/3)(y + 1/4)^{3/2} \Big|_0^2 = (4\pi/3)((9/4)^{3/2} - (1/4)^{3/2}) \\ &= 13\pi/3 \end{aligned}$$

$$4\mathbf{G-6} \quad y = (a^{2/3} - x^{2/3})^{3/2} \implies y' = -x^{-1/3}(a^{2/3} - x^{2/3})^{1/2}. \quad \text{Hence}$$

$$ds = \sqrt{1 + x^{-2/3}(a^{2/3} - x^{2/3})} dx = a^{1/3} x^{-1/3} dx$$

Therefore, (using symmetry on the interval $-a \leq x \leq a$)



$$\begin{aligned} A &= \int 2\pi y ds = 2 \int_0^a 2\pi (a^{2/3} - x^{2/3})^{3/2} a^{1/3} x^{-1/3} dx \\ &= (4\pi)(2/5)(-3/2)a^{1/3} (a^{2/3} - x^{2/3})^{5/2} \Big|_0^a \\ &= (12\pi/5)a^2 \end{aligned}$$

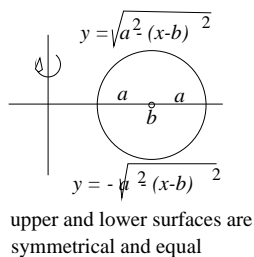
$$4\mathbf{G-7} \quad \text{a) Top half: } y = \sqrt{a^2 - (x - b)^2}, \quad y' = (b - x)/y. \quad \text{Hence,}$$

$$ds = \sqrt{1 + (b - x)^2/y^2} dx = \sqrt{(y^2 + (b - x)^2)/y^2} dx = (a/y) dx$$

Since we are only covering the top half we double the integral for area:

$$A = \int 2\pi x ds = 4\pi a \int_{b-a}^{b+a} \frac{x dx}{\sqrt{a^2 - (x - b)^2}}$$

$$\text{b) We need to rotate two curves } x_2 = b + \sqrt{a^2 - y^2}$$



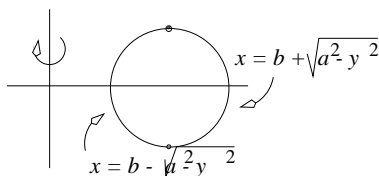
and $x_1 = b - \sqrt{a^2 - y^2}$ around the y -axis. The value

$$dx_2/dy = -(dx_1/dy) = -y/\sqrt{a^2 - y^2}$$

So in both cases,

$$ds = \sqrt{1 + y^2/(a^2 - y^2)} dy = (a/\sqrt{a^2 - y^2}) dy$$

The integral is



$$A = \int 2\pi x_2 ds + \int 2\pi x_1 ds = \int_{-a}^a 2\pi(x_1 + x_2) \frac{ady}{\sqrt{a^2 - y^2}}$$

But $x_1 + x_2 = 2b$, so

$$A = 4\pi ab \int_{-a}^a \frac{dy}{\sqrt{a^2 - y^2}}$$

c) Substitute $y = a \sin \theta$, $dy = a \cos \theta d\theta$ to get

$$A = 4\pi ab \int_{-\pi/2}^{\pi/2} \frac{a \cos \theta d\theta}{a \cos \theta} = 4\pi ab \int_{-\pi/2}^{\pi/2} d\theta = 4\pi^2 ab$$

4H. Polar coordinate graphs

4H-1 We give the polar coordinates in the form (r, θ) :

- a) $(3, \pi/2)$ b) $(2, \pi)$ c) $(2, \pi/3)$ d) $(2\sqrt{2}, 3\pi/4)$
 e) $(\sqrt{2}, -\pi/4 \text{ or } 7\pi/4)$ $(2, -\pi/2 \text{ or } 3\pi/2)$
 g) $(2, -\pi/6 \text{ or } 11\pi/6)$ $(2\sqrt{2}, -3\pi/4 \text{ or } 5\pi/4)$

4H-2 a) (i) $(x - a)^2 + y^2 = a^2 \Rightarrow x^2 - 2ax + y^2 = 0 \Rightarrow r^2 - 2ar \cos \theta = 0 \Rightarrow r = 2a \cos \theta$.

(ii) $\angle OPQ = 90^\circ$, since it is an angle inscribed in a semicircle.
 In the right triangle OPQ, $|OP| = |OQ| \cos \theta$, i.e., $r = 2a \cos \theta$.

b) (i) Analogous to 4H-2a(i); ans: $r = 2a \sin \theta$.
 (ii) analogous to 4H-2a(ii); note that $\angle OQP = \theta$, since both angles are complements of $\angle POQ$.

c) (i) OQP is a right triangle, $|OP| = r$, and $\angle POQ = \alpha - \theta$.
 The polar equation is $r \cos(\alpha - \theta) = a$, or in expanded form,

$$r(\cos \alpha \cos \theta + \sin \alpha \sin \theta) = a, \text{ or finally}$$

$$\frac{x}{A} + \frac{y}{B} = 1,$$

since from the right triangles OAQ and OBQ , we have $\cos \alpha = \frac{a}{A}$, $\sin \alpha = \cos BOQ = \frac{a}{B}$.

d) Since $|OQ| = \sin \theta$, we have:
 if P is above the x -axis, $\sin \theta > 0$, $|OP| = |OQ| - |QR|$, or $r = a - a \sin \theta$;
 if P is below the x -axis, $\sin \theta < 0$, $|OP| = |OQ| + |QR|$, or $r = a + a |\sin \theta| = a - a \sin \theta$. Thus the equation is $r = a(1 - \sin \theta)$.

e) Briefly, when $P = (0, 0)$, $|PQ||PR| = a \cdot a = a^2$, the constant.
 Using the law of cosines,

$$|PR|^2 = r^2 + a^2 - 2ar \cos \theta;$$

$$|PQ|^2 = r^2 + a^2 - 2ar \cos(\pi - \theta) = r^2 + a^2 + 2ar \cos \theta$$

Therefore

$$|PQ|^2 |PR|^2 = (r^2 + a^2)^2 - (2ar \cos \theta)^2 = (a^2)^2$$

which simplifies to

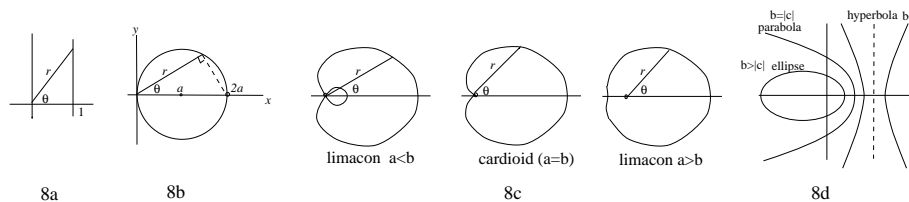
$$r^2 = 2a^2 \cos 2\theta.$$

4H-3 a) $r = \sec \theta \Rightarrow r \cos \theta = 1 \Rightarrow x = 1$

$$\text{b) } r = 2a \cos \theta \Rightarrow r^2 = r \cdot 2a \cos \theta = 2ax \Rightarrow x^2 + y^2 = 2ax$$

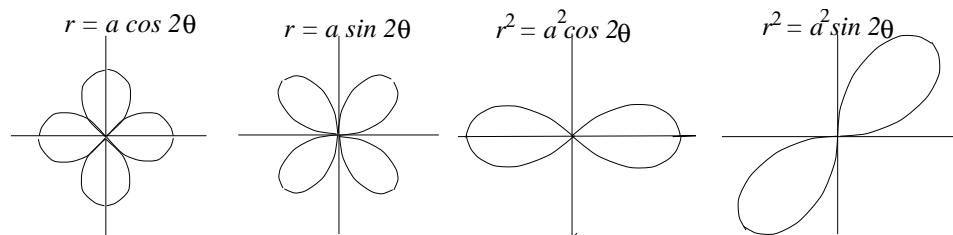
c) $r = (a + b \cos \theta)$ (This figure is a cardioid for $a = b$, a limaçon with a loop for $0 < a < b$, and a limaçon without a loop for $a > b > 0$.)

$$r^2 = ar + br \cdot \cos \theta = ar + bx \implies x^2 + y^2 = a\sqrt{x^2 + y^2} + bx$$



$$\begin{aligned} \text{(d)} \quad r = a/(b + c \cos \theta) &\implies r(b + c \cos \theta) = a \implies rb + cx = a \\ &\implies rb = a - cx \implies r^2 b^2 = a^2 - 2acx + c^2 x^2 \\ &\implies a^2 - 2acx + (c^2 - b^2)x^2 - b^2 y^2 = 0 \end{aligned}$$

$$\begin{aligned} \text{(e)} \quad r = a \sin(2\theta) &\implies r = 2a \sin \theta \cos \theta = 2axy/r^2 \\ &\implies r^3 = 2axy \implies (x^2 + y^2)^{3/2} = 2axy \end{aligned}$$



$$\text{f)} \quad r = a \cos(2\theta) = a(2 \cos^2 \theta - 1) = a\left(\frac{2x^2}{x^2 + y^2} - 1\right) \implies (x^2 + y^2)^{3/2} = a(x^2 - y^2)$$

$$\text{g)} \quad r^2 = a^2 \sin(2\theta) = 2a^2 \sin \theta \cos \theta = 2a^2 \frac{xy}{r^2} \implies r^4 = 2a^2 xy \implies (x^2 + y^2)^2 = 2axy$$

$$\text{h)} \quad r^2 = a^2 \cos(2\theta) = a^2\left(\frac{2x^2}{x^2 + y^2} - 1\right) \implies (x^2 + y^2)^2 = a^2(x^2 - y^2)$$

$$\text{i)} \quad r = e^{a\theta} \implies \ln r = a\theta \implies \ln \sqrt{x^2 + y^2} = a \tan^{-1} \frac{y}{x}$$

4I. Area and arclength in polar coordinates

$$\mathbf{4I-1} \quad \sqrt{(dr/d\theta)^2 + r^2}d\theta$$

a) $\sec^2 \theta d\theta$

b) $2ad\theta$

c) $\sqrt{a^2 + b^2 + 2ab \cos \theta}d\theta$

d) $\frac{a\sqrt{b^2 + c^2 + 2bc \cos \theta}}{(b + c \cos \theta)^2}d\theta$

e) $a\sqrt{4 \cos^2(2\theta) + \sin^2(2\theta)}d\theta$

f) $a\sqrt{4 \sin^2(2\theta) + \cos^2(2\theta)}d\theta$

g) Use implicit differentiation:

$$2rr' = 2a^2 \cos(2\theta) \implies r' = a^2 \cos(2\theta)/r \implies (r')^2 = a^2 \cos^2(2\theta)/\sin(2\theta)$$

Hence, using a common denominator and $\cos^2 + \sin^2 = 1$,

$$ds = \sqrt{a^2 \cos^2(2\theta)/\sin(2\theta) + a^2 \sin(2\theta)}d\theta = \frac{a}{\sqrt{\sin(2\theta)}}d\theta$$

h) This is similar to (g):

$$ds = \frac{a}{\sqrt{\cos(2\theta)}}d\theta$$

i) $\sqrt{1 + a^2 e^{a\theta}}d\theta$

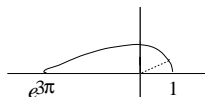
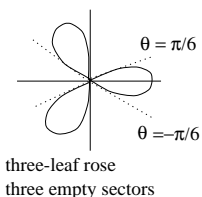
4I-2 $dA = (r^2/2)d\theta$. The main difficulty is to decide on the endpoints of integration. Endpoints are successive times when $r = 0$.

$$\cos(3\theta) = 0 \implies 3\theta = \pi/2 + k\pi \implies \theta = \pi/6 + k\pi/3, \quad k \text{ an integer.}$$

$$\text{Thus, } A = \int_{-\pi/6}^{\pi/6} (a^2 \cos^2(3\theta)/2)d\theta = a^2 \int_0^{\pi/6} \cos^2(3\theta)d\theta.$$

(Stop here in Unit 4. Evaluated in Unit 5.)

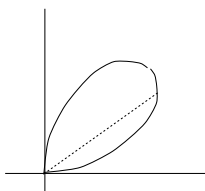
$$\mathbf{4I-3} \quad A = \int (r^2/2)d\theta = \int_0^\pi (e^{6\theta}/2)d\theta = (1/12)e^{6\theta}\Big|_0^\pi = (e^{6\pi} - 1)/12$$



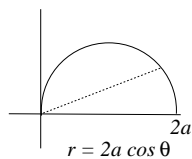
4I-4 Endpoints are successive time when $r = 0$.

$$\sin(2\theta) = 0 \implies 2\theta = k\pi, \quad k \text{ an integer.}$$

$$\text{Thus, } A = \int_0^{\pi/2} (r^2/2) d\theta = \int_0^{\pi/2} (a^2/2) \sin(2\theta) d\theta = -(a^2/4) \cos(2\theta) \Big|_0^{\pi/2} = a^2/2.$$



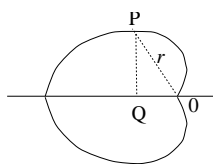
4I-5 $r = 2a \cos \theta$, $ds = 2a d\theta$, $-\pi/2 < \theta < \pi/2$. (The range was chosen carefully so that $r > 0$.) Total length of the circle is $2\pi a$. Since the upper and lower semicircles are symmetric, it suffices to calculate the average over the upper semicircle:



$$\frac{1}{\pi a} \int_0^{\pi/2} 2a \cos \theta (2a) d\theta = \frac{4a}{\pi} \sin \theta \Big|_0^{\pi/2} = \frac{4a}{\pi}$$

4I-6 a) Since the upper and lower halves of the cardioid are symmetric, it suffices to calculate the average distance to the x-axis just for a point on the upper half. We have $r = a(1 - \cos \theta)$, and the distance to the x-axis is $r \sin \theta$, so

$$\frac{1}{\pi} \int_0^{\pi} r \sin \theta d\theta = \frac{1}{\pi} \int_0^{\pi} a(1 - \cos \theta) \sin \theta d\theta = \frac{a}{2\pi} (1 - \cos \theta)^2 \Big|_0^{\pi} = \frac{2a}{\pi}$$



$$(b) \quad ds = \sqrt{(dr/d\theta)^2 + r^2} d\theta = a\sqrt{(1 - \cos\theta)^2 + \sin^2\theta} d\theta \\ = a\sqrt{2 - 2\cos\theta} d\theta = 2a \sin(\theta/2) d\theta, \quad \text{using the half angle formula.}$$

$$\text{arclength} = \int_0^{2\pi} 2a \sin(\theta/2) d\theta = -4a \cos(\theta/2) \Big|_0^{2\pi} = 8a$$

For the average, don't use the half-angle version of the formula for ds , and use the interval $-\pi < \theta < \pi$, where $\sin\theta$ is odd:

$$\text{Average} = \frac{1}{8a} \int_{-\pi}^{\pi} |r \sin\theta| a \sqrt{2 - 2\cos\theta} d\theta = \frac{1}{8a} \int_{-\pi}^{\pi} |\sin\theta| \sqrt{2} a^2 (1 - \cos\theta)^{3/2} d\theta \\ = \frac{\sqrt{2}a}{4} \int_0^{\pi} (1 - \cos\theta)^{3/2} \sin\theta d\theta = \frac{\sqrt{2}a}{10} (1 - \cos\theta)^{5/2} \Big|_0^{\pi} = \frac{4}{5}a$$

4I-7 $dx = -a \sin\theta d\theta$. So the semicircle $y > 0$ has area

$$\int_{-a}^a y dx = \int_{\pi}^0 a \sin\theta (-a \sin\theta) d\theta = a^2 \int_0^{\pi} \sin^2\theta d\theta$$

But

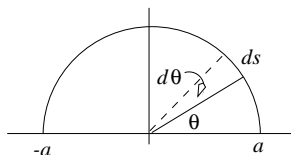
$$\int_0^{\pi} \sin^2\theta d\theta = \frac{1}{2} \int_0^{\pi} (1 - \cos(2\theta)) d\theta = \pi/2$$

So the area is $\pi a^2/2$ as it should be for a semicircle.

$$\text{Arclength: } ds^2 = dx^2 + dy^2$$

$$\implies (ds)^2 = (-a \sin\theta d\theta)^2 + (a \cos\theta d\theta)^2 = a^2(\sin^2\theta + \cos^2\theta)(d\theta)^2$$

$$\implies ds = a d\theta \quad (\text{obvious from picture}).$$



$$\int ds = \int_0^{2\pi} a d\theta = 2\pi a$$

4J. Other applications

4J-1 Divide the water in the hole into n equal circular discs of thickness Δy .

Volume of each disc: $\pi \left(\frac{1}{2}\right)^2 \Delta y$

Energy to raise the disc of water at depth y_i to surface: $\frac{\pi}{4} k y_i \Delta y$.

Adding up the energies for the different discs, and passing to the limit,

$$E = \lim_{n \rightarrow \infty} \sum_1^n \frac{\pi}{4} k y_i \Delta y = \int_0^{100} \frac{\pi}{4} k y \, dy = \left. \frac{\pi k y^2}{4 \cdot 2} \right|_0^{100} = \frac{\pi k 10^4}{8}.$$

4J-2 Divide the hour into n equal small time intervals Δt .

At time t_i , $i = 1, \dots, n$, there are $x_0 e^{-kt_i}$ grams of material, producing approximately $r x_0 e^{-kt_i} \Delta t$ radiation units over the time interval $[t_i, t_i + \Delta t]$.

Adding and passing to the limit,

$$R = \lim_{n \rightarrow \infty} \sum_1^n r x_0 e^{-kt_i} \Delta t = \int_0^{60} r x_0 e^{-kt} \, dt = r x_0 \left. \frac{e^{-kt}}{-k} \right|_0^{60} = \frac{r x_0}{k} (1 - e^{-60k}).$$

4J-3 Divide up the pool into n thin concentric cylindrical shells, of radius r_i , $i = 1, \dots, n$, and thickness Δr .

The volume of the i -th shell is approximately $2\pi r_i D \Delta r$.

The amount of chemical in the i -th shell is approximately $\frac{k}{1+r_i^2} 2\pi r_i D \Delta r$.

Adding, and passing to the limit,

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} \sum_1^n \frac{k}{1+r_i^2} 2\pi r_i D \Delta r = \int_0^R 2\pi k D \frac{r}{1+r^2} \, dr \\ &= \left. \pi k D \ln(1+r^2) \right|_0^R = \pi k D \ln(1+R^2) \text{ gms.} \end{aligned}$$

4J-4 Divide the time interval into n equal small intervals of length Δt by the points t_i , $i = 1, \dots, n$.

The approximate number of heating units required to maintain the temperature at 75° over the time interval $[t_i, t_i + \Delta t]$: is

$$\left[75 - 10 \left(6 - \cos \frac{\pi t_i}{12} \right) \right] \cdot k \Delta t.$$

Adding over the time intervals and passing to the limit:

$$\begin{aligned} \text{total heat} &= \lim_{n \rightarrow \infty} \sum_1^n \left[75 - 10 \left(6 - \cos \frac{\pi t_i}{12} \right) \right] \cdot k \Delta t \\ &= \int_0^{24} k \left[75 - 10 \left(6 - \cos \frac{\pi t}{12} \right) \right] dt \\ &= \int_0^{24} k \left(15 + 10 \cos \frac{\pi t}{12} \right) dt = k \left[15t + \frac{120}{\pi} \sin \frac{\pi t}{12} \right]_0^{24} = 360k. \end{aligned}$$

4J-5 Divide the month into n equal intervals of length Δt by the points t_i , $i = 1, \dots, n$.

Over the time interval $[t_i, t_i + \Delta t]$, the number of units produced is about $(10 + t_i) \Delta t$.

The cost of holding these in inventory until the end of the month is $c(30 - t_i)(10 + t_i) \Delta t$.

Adding and passing to the limit,

$$\begin{aligned} \text{total cost} &= \lim_{n \rightarrow \infty} \sum_1^n c(30 - t_i)(10 + t_i) \Delta t \\ &= \int_0^{30} c(30 - t)(10 + t) dt = c \left[300t + 10t^2 - \frac{t^3}{3} \right]_0^{30} = 9000c. \end{aligned}$$

4J-6 Divide the water in the tank into thin horizontal slices of width dy .

If the slice is at height y above the center of the tank, its radius is $\sqrt{r^2 - y^2}$. This formula for the radius of the slice is correct even if $y < 0$ – i.e., the slice is below the center of the tank – as long as $-r < y < r$, so that there really is a slice at that height.

Volume of water in the slice = $\pi(r^2 - y^2) dy$

Weight of water in the slice = $\pi w(r^2 - y^2) dy$

Work to lift this slice from the ground to the height $h + y$ = $\pi w(r^2 - y^2) dy (h + y)$.

$$\begin{aligned} \text{Total work} &= \int_{-r}^r \pi w(r^2 - y^2)(h + y) dy \\ &= \pi w \int_{-r}^r (r^2 h + r^2 y - h y^2 - y^3) dy \\ &= \pi w \left[r^2 h y + \frac{r^2 y^2}{2} - \frac{h y^3}{3} - \frac{y^4}{4} \right]_{-r}^r. \end{aligned}$$

In this last line, the even powers of y have the same value at $-r$ and r , so contribute 0 when it is evaluated; we get therefore

$$= \pi w h \left[r^2 y - \frac{y^3}{3} \right]_{-r}^r = 2\pi w h \left(r^3 - \frac{r^3}{3} \right) = \frac{4}{3} \pi w h r^3.$$

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