GLOBAL EDITION

Feedback Control of Dynamic Systems

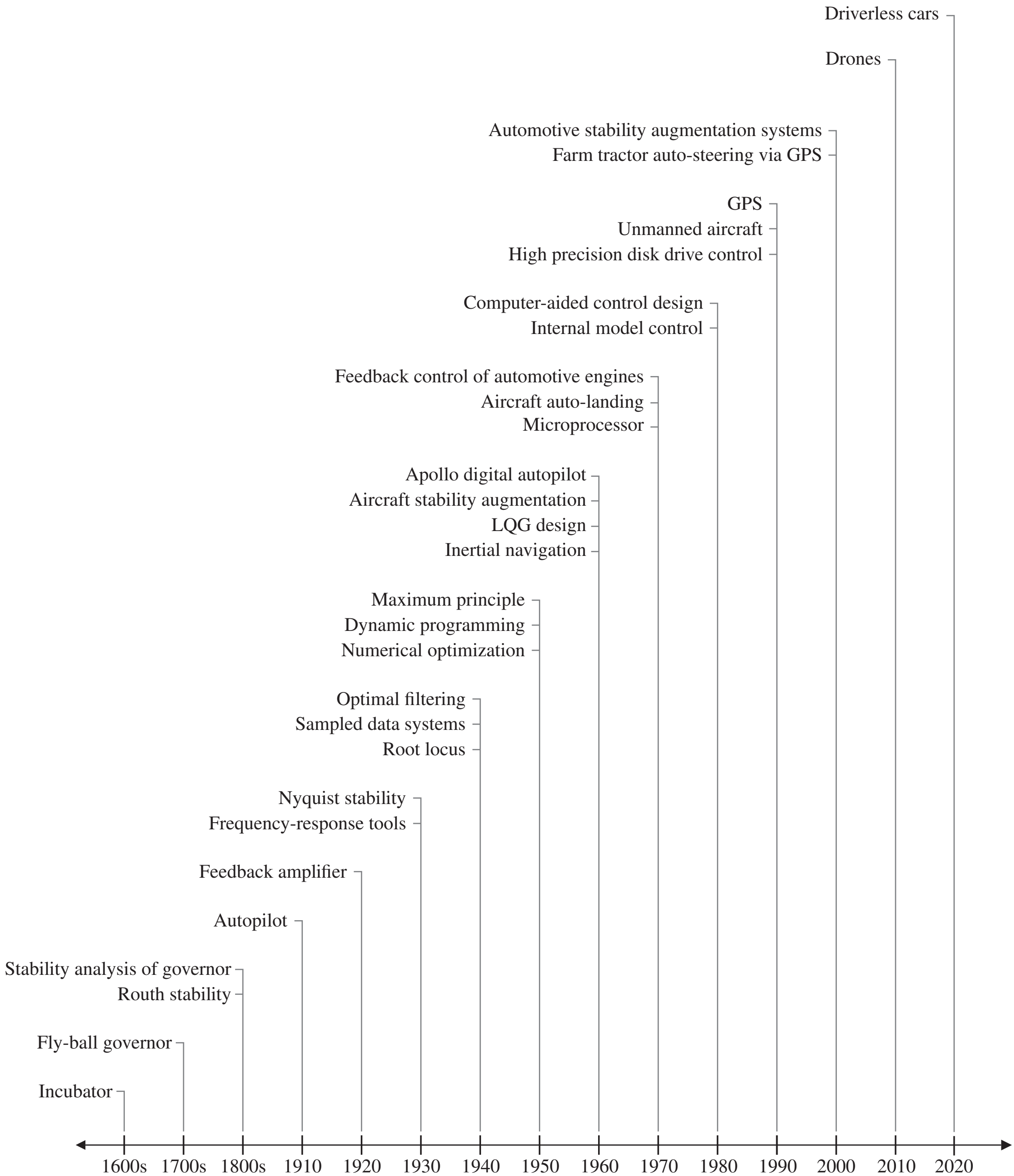
EIGHTH EDITION

Franklin • Powell • Emami-Naeini

Table of Laplace Transforms

Number	$F(s)$	$f(t), t \geq 0$
1	1	$\delta(t)$
2	$\frac{1}{s}$	$1(t)$
3	$\frac{1}{s^2}$	t
4	$\frac{2!}{s^3}$	t^2
5	$\frac{3!}{s^4}$	t^3
6	$\frac{m!}{s^{m+1}}$	t^m
7	$\frac{1}{(s+a)}$	e^{-at}
8	$\frac{1}{(s+a)^2}$	te^{-at}
9	$\frac{1}{(s+a)^3}$	$\frac{1}{2!}t^2e^{-at}$
10	$\frac{1}{(s+a)^m}$	$\frac{1}{(m-1)!}t^{m-1}e^{-at}$
11	$\frac{a}{s(s+a)}$	$1-e^{-at}$
12	$\frac{a}{s^2(s+a)}$	$\frac{1}{a}(at-1+e^{-at})$
13	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$
14	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$
15	$\frac{a^2}{s(s+a)^2}$	$1-e^{-at}(1+at)$
16	$\frac{(b-a)s}{(s+a)(s+b)}$	$be^{-bt}-ae^{-at}$
17	$\frac{a}{(s^2+a^2)}$	$\sin at$
18	$\frac{s}{(s^2+a^2)}$	$\cos at$
19	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$
20	$\frac{b}{(s+a)^2+b^2}$	$e^{-at}\sin bt$
21	$\frac{a^2+b^2}{s\left[(s+a)^2+b^2\right]}$	$1-e^{-at}\left(\cos bt+\frac{a}{b}\sin bt\right)$

Chronological History of Feedback Control



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Feedback Control of Dynamic Systems

Eighth Edition

Global Edition

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To Valerie, Daisy, Annika, Davenport, Malahat, Sheila, Nima, and to the memory of Gene

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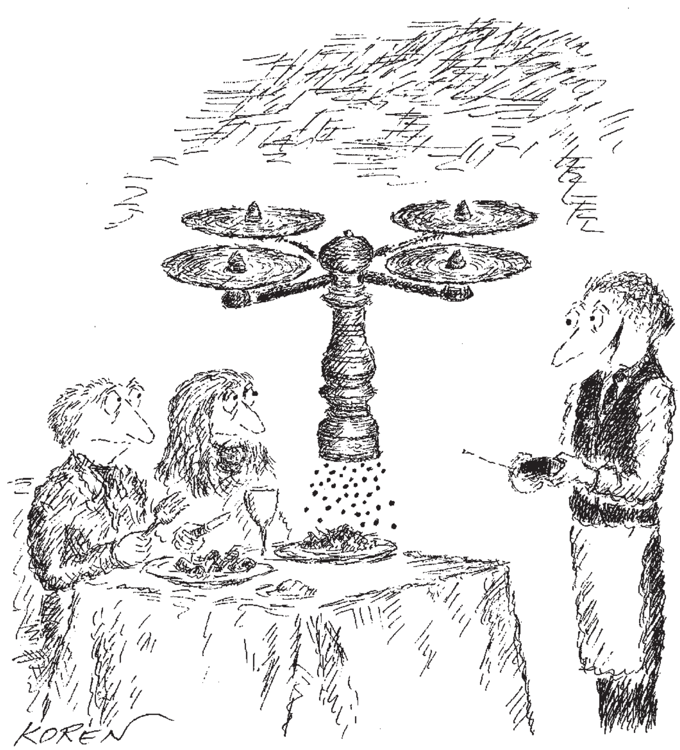
Preface

In this Eighth Edition we again present a text in support of a first course in control and have retained the best features of our earlier editions. For this edition, we have responded to a survey of users by adding some new material (for example, drone dynamics and control) and deleted other little-used material from the book. We have also updated the text throughout so that it uses the improved features of MATLAB®. Drones have been discussed extensively in the controls literature as well as the common press. They are being used in mining, construction, aerial photography, search and rescue, movie industry, package delivery, mapping, surveying, farming, animal research, hurricane hunting, and defense. Since feedback control is a necessary component of all the drones, we develop the equations of motion in Chapter 2, and follow that with control design examples in the chapters 5, 6, 7, and 10. They have great potential for many tasks and could speed up and lessen the cost of these activities. The figure below symbolizes the widespread interest in this exciting new field.



"Fresh pepper?"

Source: Edward Koren/ The New Yorker © Conde Nast

The basic structure of the book is unchanged and we continue to combine analysis with design using the three approaches of the root locus, frequency response, and state-variable equations. The text continues to include many carefully worked out examples to illustrate the material. As before, we provide a set of review questions at the end of each chapter with answers in the back of the book to assist the students in verifying that they have learned the material.

In the three central chapters on design methods we continue to expect the students to learn how to perform the very basic calculations by hand and make a rough sketch of a root locus or Bode plot as a sanity check on the computer results and as an aid to design. However, we introduce the use of Matlab early on in recognition of the universal use of software tools in control analysis and design. As before, we have prepared a collection of all the Matlab files (both "m"files and SIMULINK® "slx" files) used to produce the figures in the book. These are available along with the advanced material described above at our website at *www.pearsonglobaleditions.com*.

New to this Edition

We feel that this Eighth Edition presents the material with good pedagogical support, provides strong motivation for the study of control, and represents a solid foundation for meeting the educational challenges. We introduce the study of feedback control, both as a specialty of itself and as support for many other fields.

A more detailed list of the changes is:

- Deleted the disk drive and tape drive examples from Chapters 2, 7, and 10
- Added drone examples and/or problems in Chapters 2, 5, 6, 7, and 10
- Added a thermal system control example to Chapters 2 and 4
- Added a section on anti-windup for integral control in Chapter 9
- Added Cramer's Rule to chapter 2 and Appendix WB
- Updated Matlab commands throughout the book and in Appendix C
- Updated the section on PID tuning in chapter 4
- Updated the engine control and chemotaxis case studies in Chapter 10
- Over 60 of the problems in this edition are either new or revised from the 7th edition

Addressing the Educational Challenges

Some of the educational challenges facing students of feedback control are long-standing; others have emerged in recent years. Some of the challenges remain for students across their entire engineering education; others are unique to this relatively sophisticated course. Whether they are old or new, general or particular, the educational challenges we perceived were critical to the evolution of this text. Here, we will state several educational challenges and describe our approaches to each of them.

- CHALLENGE *Students must master design as well as analysis techniques.*

Design is central to all of engineering and especially so to control systems. Students find that design issues, with their corresponding opportunities to tackle practical applications, are particularly motivating. But students also find design problems difficult because design problem statements are usually poorly posed and lack unique solutions. Because of both its inherent importance and its motivational effect on students, design is emphasized throughout this text so confidence in solving design problems is developed from the start.

The emphasis on design begins in Chapter 4 following the development of modeling and dynamic response. The basic idea of feedback is introduced first, showing its influence on disturbance rejection, tracking accuracy, and robustness to parameter changes. The design orientation continues with uniform treatments of the root locus, frequency response, and state variable feedback techniques. All the treatments are aimed at providing the knowledge necessary to find a good feedback control design with no more complex mathematical development than is essential to clear understanding.

Throughout the text, examples are used to compare and contrast the design techniques afforded by the different design methods and, in the capstone case studies of Chapter 10, complex real-world design problems are attacked using all the methods in a unified way.

- CHALLENGE *New ideas continue to be introduced into control.*

Control is an active field of research and hence there is a steady influx of new concepts, ideas, and techniques. In time, some of these elements develop to the point where they join the list of things every control engineer must know. This text is devoted to supporting students equally in their need to grasp both traditional and more modern topics.

In each of our editions, we have tried to give equal importance to root locus, frequency response, and state-variable methods for design. In this edition, we continue to emphasize solid mastery of the underlying techniques, coupled with computer-based methods for detailed calculation. We also provide an early introduction to data sampling and discrete controllers in recognition of the major role played by digital controllers in our field. While this material can be skipped to save time without harm to the flow of the text, we feel that it is very important for students to understand that computer control is widely used and that the most basic techniques of computer control are easily mastered.

- CHALLENGE *Students need to manage a great deal of information.*

The vast array of systems to which feedback control is applied and the growing variety of techniques available for the solution of control problems means that today's student of feedback control must learn many new ideas. How do students keep their perspective as they plow through lengthy and complex textual passages? How do they identify highlights and draw appropriate conclusions? How do they review for exams? Helping students with these tasks was a criterion for the Fourth, Fifth, Sixth, and Seventh Editions and continues to be addressed in this Eighth Edition. We outline these features below.

FEATURE

1. *Chapter openers* offer perspective and overview. They place the specific chapter topic in the context of the discipline as a whole, and they briefly overview the chapter sections.
2. *Margin notes* help students scan for chapter highlights. They point to important definitions, equations, and concepts.
3. *Shaded highlights* identify key concepts within the running text. They also function to summarize important design procedures.
4. *Bulleted chapter summaries* help with student review and prioritization. These summaries briefly reiterate the key concepts and conclusions of the chapter.
5. *Synopsis of design aids*. Relationships used in design and throughout the book are collected inside the back cover for easy reference.
6. *The color blue* is used (1) to highlight useful pedagogical features, (2) to highlight components under particular scrutiny within block diagrams, (3) to distinguish curves on graphs, and (4) to lend a more realistic look to figures of physical systems.
7. *Review questions* at the end of each chapter with solutions in the back to guide the student in self-study
8. *Historical perspectives* at the end of each chapter provide some background and color on how or why the material in that particular chapter evolved.

- CHALLENGE *Students of feedback control come from a wide range of disciplines*.

Feedback control is an interdisciplinary field in that control is applied to systems in every conceivable area of engineering. Consequently, some schools have separate introductory courses for control within the standard disciplines and some, such as Stanford, have a single set of courses taken by students from many disciplines. However, to restrict the examples to one field is to miss much of the range and power of feedback but to cover the whole range of applications is overwhelming. In this book, we develop the interdisciplinary nature of the field and provide review material for several of the most common technologies so that students from many disciplines will be comfortable with the presentation. For Electrical Engineering students who typically have a good background in transform analysis, we include in Chapter 2 an introduction to writing equations of motion for mechanical mechanisms. For mechanical engineers, we include in Chapter 3 a review of the Laplace transform and dynamic response as needed in control. In addition, we introduce other technologies briefly and, from time to time, we present the equations of motion of a physical system without derivation but with enough physical description to be understood from a response point of view. Examples of some of the physical systems represented in the text include a quadrotor drone, a satellite tracking system, the fuel–air ratio in an automobile engine, and an airplane automatic pilot system.

Outline of the Book

The contents of the printed book are organized into ten chapters and three appendices. Optional sections of advanced or enrichment material marked with a triangle (△) are included at the end of some chapters. Examples and problems based on this material are also marked with a triangle (△). There are also four full appendices on the website plus numerous appendices that supplement the material in most of the chapters. The appendices in the printed book include Laplace transform tables, answers to the end-of-chapter review questions, and a list of Matlab commands. The appendices on the website include a review of complex variables, a review of matrix theory, some important results related to state-space design, and optional material supporting or extending several of the chapters.

In Chapter 1, the essential ideas of feedback and some of the key design issues are introduced. This chapter also contains a brief history of control, from the ancient beginnings of process control to flight control and electronic feedback amplifiers. It is hoped that this brief history will give a context for the field, introduce some of the key people who contributed to its development, and provide motivation to the student for the studies to come.

Chapter 2 is a short presentation of dynamic modeling and includes mechanical, electrical, electromechanical, fluid, and thermodynamic devices. This material can be omitted, used as the basis of review homework to smooth out the usual nonuniform preparation of students, or covered in-depth depending on the needs of the students.

Chapter 3 covers dynamic response as used in control. Again, much of this material may have been covered previously, especially by electrical engineering students. For many students, the correlation between pole locations and transient response and the effects of extra zeros and poles on dynamic response represent new material. Stability of dynamic systems is also introduced in this chapter. This material needs to be covered carefully.

Chapter 4 presents the basic equations and transfer functions of feedback along with the definitions of the sensitivity function. With these tools, open-loop and closed-loop control are compared with respect to disturbance rejection, tracking accuracy, and sensitivity to model errors. Classification of systems according to their ability to track polynomial reference signals or to reject polynomial disturbances is described with the concept of system type. Finally, the classical proportional, integral, and derivative (PID) control structure is introduced and the influence of the controller parameters on a system's characteristic equation is explored along with PID tuning methods.

Following the overview of feedback in Chapter 4, the core of the book presents the design methods based on root locus, frequency response, and state-variable feedback in Chapters 5, 6, and 7, respectively.

Chapter 8 develops the tools needed to design feedback control for implementation in a digital computer. However, for a complete treatment of feedback control using digital computers, the reader is referred to the companion text, *Digital Control of Dynamic Systems*, by Franklin, Powell, and Workman; Ellis-Kagle Press, 1998.

In Chapter 9, the nonlinear material includes techniques for the linearization of equations of motion, analysis of zero memory nonlinearity as a variable gain, frequency response as a describing function, the phase plane, Lyapunov stability theory, and the circle stability criterion.

In Chapter 10, the three primary approaches are integrated in several case studies, and a framework for design is described that includes a touch of the real-world context of practical control design.

Course Configurations

The material in this text can be covered flexibly. Most first-course students in controls will have some dynamics and Laplace transforms. Therefore, Chapter 2 and most of Chapter 3 would be a review for those students. In a ten-week quarter, it is possible to review Chapter 3, and cover all of Chapters 1, 4, 5, and 6. Most optional sections should be omitted. In the second quarter, Chapters 7 and 9 can be covered comfortably including the optional sections. Alternatively, some optional sections could be omitted and selected portions of Chapter 8 included. A semester course should comfortably accommodate Chapters 1–7, including the review materials of Chapters 2 and 3, if needed. If time remains after this core coverage, some introduction of digital control from Chapter 8, selected nonlinear issues from Chapter 9, and some of the case studies from Chapter 10 may be added.

The entire book can also be used for a three-quarter sequence of courses consisting of modeling and dynamic response (Chapters 2 and 3), classical control (Chapters 4–6), and modern control (Chapters 7–10).

Two basic 10-week courses are offered at Stanford and are taken by seniors and first-year graduate students who have not had a course in control, mostly in the departments of Aeronautics and Astronautics, Mechanical Engineering, and Electrical Engineering. The first course reviews Chapters 2 and 3 and covers Chapters 4–6. The more advanced course is intended for graduate students and reviews Chapters 4–6 and covers Chapters 7–10. This sequence complements a graduate course in linear systems and is the prerequisite to courses in digital control, nonlinear control, optimal control, flight control, and smart product design. Some of the subsequent courses include extensive laboratory experiments. Prerequisites for the course sequence include dynamics or circuit analysis and Laplace transforms.

Prerequisites to This Feedback Control Course

This book is for a first course at the senior level for all engineering majors. For the core topics in Chapters 4–7, prerequisite understanding of modeling and dynamic response is necessary. Many students will come into the course with sufficient background in those concepts from previous courses in physics, circuits, and dynamic response. For those needing review, Chapters 2 and 3 should fill in the gaps.

An elementary understanding of matrix algebra is necessary to understand the state-space material. While all students will have much of this in prerequisite math courses, a review of the basic relations is given in online Appendix WB and a brief treatment of particular material needed in control is given at the start of Chapter 7. The emphasis is on the relations between linear dynamic systems and linear algebra.

Supplements

The website *www.pearsonglobaleditions.com* includes the dot-m and dot-slx files used to generate all the Matlab figures in the book, and these may be copied and distributed to the students as desired. The websites also contain some more advanced material and appendices which are outlined in the Table of Contents. A Solutions Manual with complete solutions to all homework problems is available to instructors only.

Acknowledgments

Finally, we wish to acknowledge our great debt to all those who have contributed to the development of feedback control into the exciting field it is today and specifically to the considerable help and education we have received from our students and our colleagues. In particular, we have benefited in this effort by many discussions with the following

who taught introductory control at Stanford: A. E. Bryson, Jr., R. H. Cannon, Jr., D. B. DeBra, S. Rock, S. Boyd, C. Tomlin, P. Enge, A. Okamura, and C. Gerdes. Other colleagues who have helped us include D. Fraser, N. C. Emami, B. Silver, M. Dorfman, K. Rudie, L. Pao, F. Khorrami, K. Lorell, M. Tischler, D. de Roover, R. Patrick, M. Berrios, J. K. Lee, J. L. Ebert, I. Kroo, K. Leung, and M. Schwager. Special thanks go to the many students who have provided almost all the solutions to the problems in the book.

We especially want to express our great appreciation for the contributions to the book by Gene Franklin. Gene was a mentor, teacher, advisor, and good friend to us both. We had many meetings as we collaborated on earlier editions of the book over the last 28 years of his life, and every single one of those meetings has been friendly and enjoyable as we meshed our views on how to present the material. We learned control along with humor from Gene in grad school classes, and we benefitted from his mentoring: in one case as a new assistant professor, and in the other as a Ph.D. advisee. Collectively, we collaborated on research, created new courses and laboratories, and written two textbooks over a period of 40 years. Gene always had a smile with a twinkle in his eye, and was a pleasure to work with; he was a true gentleman.

J.D.P.
A.E.-N.
Stanford, California

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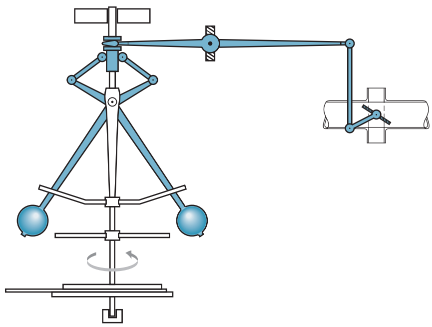
1

An Overview and Brief History of Feedback Control



A Perspective on Feedback Control

Feedback control of dynamic systems is a very old concept with many characteristics that have evolved over time. The central idea is that a dynamic system's output can be measured and fed back to a controller of some kind then used to affect the system. There are several variations on this theme.

A system that involves a person controlling a machine, as in driving an automobile, is called **manual** control. A system that involves machines only, as when room temperature can be set by a thermostat, is called **automatic** control. Systems designed to hold an output steady against unknown disturbances are called **regulators**, while systems designed to track a reference signal are called **tracking** or **servo** systems. Control systems are also classified according to the information used to compute the controlling action. If the controller does *not* use a measure of the system output being controlled in computing the control action to take, the system is called **open-loop** control. If the controlled output signal *is* measured and fed back for use in the control computation, the system is called **closed-loop** or **feedback** control. There are many other important properties of control systems in addition to these most basic characteristics. For example, we will mainly consider feedback of current measurements as opposed to predictions of the future; however, a very familiar example illustrates the limitation imposed by that assumption. When driving a car, the use of simple feedback corresponds to driving in a thick fog where one can *only see the road immediately at the front of the car* and is unable to see the future required position! Looking at the road ahead is a form of predictive control and this information, which has obvious advantages, would always be used where it is available. In most automatic control situations studied in this book, observation of the future track or disturbance is not possible. In any case, the control designer should study the process to see if any information could anticipate either a track to be followed or a disturbance to be rejected. If such a possibility is feasible, the control designer should use it to **feedforward** an early warning to the control system. An example of this is in the control of steam pressure in the boiler of an electric power generation plant. The electricity demand cycle over a day is well known; therefore, when it is known that there will soon be an increased need for electrical power, that information can be fed forward to the boiler controller in anticipation of a soon-to-be-demanded increase in steam flow.

The applications of feedback control have never been more exciting than they are today. Feedback control is an essential element in aircraft of all types: most manned aircraft, and all unmanned aircraft from large military aircraft to small drones. The FAA has predicted that the number of drones registered in the U.S. will reach 7 million by 2020! Automatic landing and collision avoidance systems in airliners are now being used routinely, and the use of satellite navigation in future designs promises a revolution in our ability to navigate aircraft in an ever more crowded airspace. The use of feedback control in driverless cars is an essential element to their success. They are now under extensive development, and predictions have been made that driverless cars will ultimately reduce the number of cars on the road by a very large percentage. The use of feedback control in surgical robotic systems is also emerging. Control is essential to the operation of systems from cell phones to jumbo jets and from washing machines to oil refineries as large as a small city. The list goes on and on. In fact, many engineers refer to control as a *hidden technology* because of its essential importance to so many devices and systems while being mainly out of sight. The future will no doubt see engineers create even more imaginative applications of feedback control.

Chapter Overview

In this chapter, we begin our exploration of feedback control using a simple familiar example: a household furnace controlled by a thermostat. The generic components of a control system are identified within the context of this example. In another example in Section 1.2—an automobile cruise control—we will develop the elementary static equations and assign numerical values to elements of the system model in order to compare the performance of open-loop control to that of feedback control when dynamics are ignored. Section 1.3 then introduces the key elements in control system design. In order to provide a context for our studies, and to give you a glimpse of how the field has evolved, Section 1.4 provides a brief history of control theory and design. In addition, later chapters have brief sections of additional historical notes on the topics covered there. Finally, Section 1.5 provides a brief overview of the contents and organization of the entire book.

1.1 A Simple Feedback System

In feedback systems, the variable being controlled—such as temperature or speed—is measured by a sensor and the measured information is fed back to the controller to influence the controlled variable. The principle is readily illustrated by a very common system, the household furnace controlled by a thermostat. The components of this system and their interconnections are shown in Fig. 1.1. Such an illustration identifies the major parts of the system and shows the directions of information flow from one component to another.

We can easily analyze the operation of this system qualitatively from the graph. Suppose both the temperature in the room where the thermostat is located and the outside temperature are significantly below the reference temperature (also called the setpoint) when power is applied. The thermostat will be *on* and the control logic will open the furnace gas valve and light the fire box. This will cause heat Q_{in} to be supplied to the house at a rate that will be significantly larger than the heat loss Q_{out}. As a result, the room temperature will rise until it exceeds the thermostat reference setting by a small amount. At this time, the furnace will be turned off and the room temperature will start to fall toward the outside value. When it falls a small amount below the setpoint,[1] the thermostat will come on again and the cycle will repeat. Typical plots of room temperature along with the furnace cycles of on and off are shown in Fig. 1.1. The outside temperature remains at 50°F and the thermostat is initially set at 55°F. At 6 a.m., the thermostat is stepped to 65°F and the furnace brings it to that level and cycles the temperature around that value thereafter. Notice the house is well insulated, so the fall of temperature with the furnace off is significantly slower than the rise with the furnace on. From this example, we can identify the generic components of the elementary feedback control system, as shown in Fig. 1.2.

The central component of this feedback system is the **process** whose output is to be controlled. In our example the process would be the house whose output is the room temperature and the **disturbance** to

[1] The **setpoint**, **reference**, and **desired input** are all the same thing and shown in Figs. 1.1–1.3.

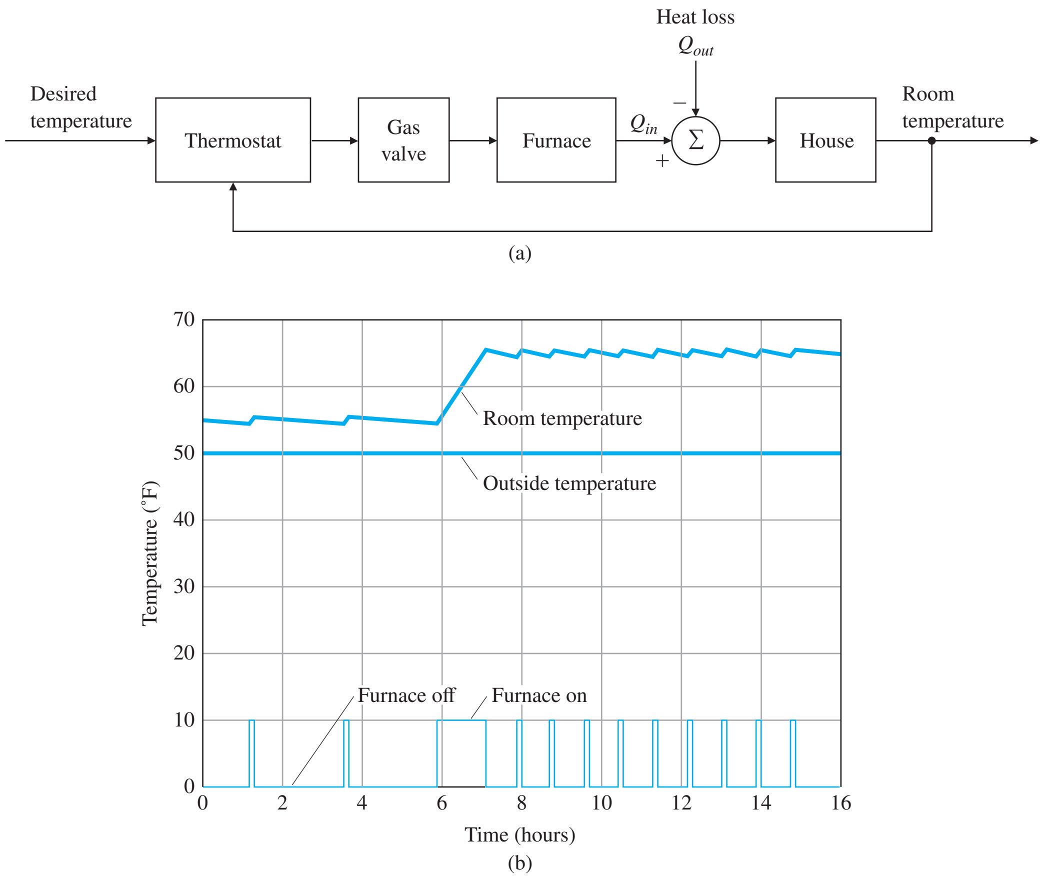


Figure 1.1

Feedback control: (a) component block diagram of a room temperature control system; (b) plot of room temperature and furnace action

the process is the flow of heat from the house, Q_{out}, due to conduction through the walls and roof to the lower outside temperature. (The outward flow of heat also depends on other factors such as wind, open doors, and so on.) The design of the process can obviously have a major impact on the effectiveness of the controls. The temperature of a well-insulated house with thermopane windows is clearly easier to control than otherwise. Similarly, the design of aircraft with control in mind makes a world of difference to the final performance. In every case, the earlier the concepts of control are introduced into the process design, the better. The **actuator** is the device that can influence the controlled variable of the process. In our case, the actuator is a gas furnace. Actually, the furnace usually has a pilot light or striking mechanism, a gas valve, and a blower fan, which turns on or off depending on the air temperature in the furnace. These details illustrate the fact that many feedback systems contain components that themselves

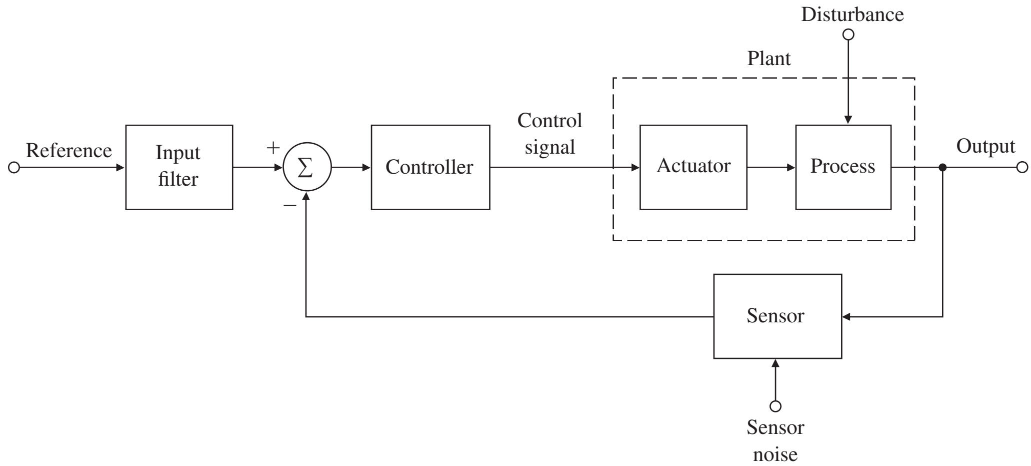


Figure 1.2

Component block diagram of an elementary feedback control

form other feedback systems.[2] The central issue with the actuator is its ability to move the process output with adequate speed and range. The furnace must produce more heat than the house loses on the worst day, and must distribute it quickly if the house temperature is to be kept in a narrow range. Power, speed, and reliability are usually more important than accuracy. Generally, the process and the actuator are intimately connected and the control design centers on finding a suitable input or control signal to send to the actuator. The combination of process and actuator is called the **plant**, and the component that actually computes the desired control signal is the **controller**. Because of the flexibility of electrical signal processing, the controller typically works on electrical signals, although the use of pneumatic controllers based on compressed air has a long and important place in process control. With the development of digital technology, cost-effectiveness and flexibility have led to the use of digital signal processors as the controller in an increasing number of cases. The component labeled **thermostat** in Fig. 1.1 measures the room temperature and is called the **sensor** in Fig. 1.2, a device whose output inevitably contains sensor noise. Sensor selection and placement are very important in control design, for it is sometimes not possible for the true controlled variable and the sensed variable to be the same. For example, although we may really wish to control the house temperature as a whole, the thermostat is in one particular room, which may or may not be at the same temperature as the rest of the house. For instance, if the thermostat is set to 68°F but is placed in the living room near a roaring fireplace, a person working in

[2] Jonathan Swift (1733) said it this way: “So, Naturalists observe, a flea Hath smaller fleas that on him prey; And these have smaller still to bite ‘em; And so proceed, *ad infinitum*.” Swift, J., On Poetry: A Rhapsody, 1733, J. Bartlett, ed., Familiar Quotations, 15th ed., Boston: Little Brown, 1980.

the study could still feel uncomfortably cold.[3,4] As we will see, in addition to placement, important properties of a sensor are the accuracy of the measurements as well as low noise, reliability, and linearity. The sensor will typically convert the physical variable into an electrical signal for use by the controller. Our general system also includes an **input filter** whose role is to convert the reference signal to electrical form for later manipulation by the controller. In some cases, the input filter can modify the reference command input in ways that improve the system response. Finally, there is a **controller** to compute the difference between the reference signal and the sensor output to give the controller a measure of the system error. The thermostat on the wall includes the sensor, input filter, and the controller. A few decades ago, the user simply set the thermostat manually to achieve the desired room temperature at the thermostat location. Over the last few decades, the addition of a small computer in the thermostat has enabled storing the desired temperature over the day and week and more recently, thermostats have gained the ability to learn what the desired temperature should be and to base that value, in part, on whether anybody will be home soon! A thermostat system that includes a motion detector can determine whether anybody is home and learns from the patterns observed what the desired temperature profile should be. The process of learning the desired setpoint is an example of artificial intelligence (AI) or machine learning, which is gaining acceptance in many fields as the power and affordability of computers improve. The combination of feedback control, AI, sensor fusion, and logic to tie it all together will become an essential feature in many future devices such as drones, driverless cars, and many others.

This text will present methods for analyzing feedback control systems and will describe the most important design techniques engineers can use in applying feedback to solve control problems. We will also study the specific advantages of feedback that compensate for the additional complexity it demands.

1.2 A First Analysis of Feedback

The value of feedback can be readily demonstrated by quantitative analysis of a simplified model of a familiar system, the cruise control of an automobile (see Fig. 1.3). To study this situation analytically, we

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[3]In the renovations of the kitchen in the house of one of the authors, the new ovens were placed against the wall where the thermostat was mounted on the other side. Now when dinner is baked in the kitchen on a cold day, the author freezes in his study unless the thermostat is reset.

[4]The story is told of the new employee at the nitroglycerin factory who was to control the temperature of a critical part of the process manually. He was told to “keep that reading below 300°.” On a routine inspection tour, the supervisor realized that the batch was dangerously hot and found the worker holding the thermometer under cold water tap to bring it down to 300°. They got out just before the explosion. Moral: sometimes automatic control is better than manual.

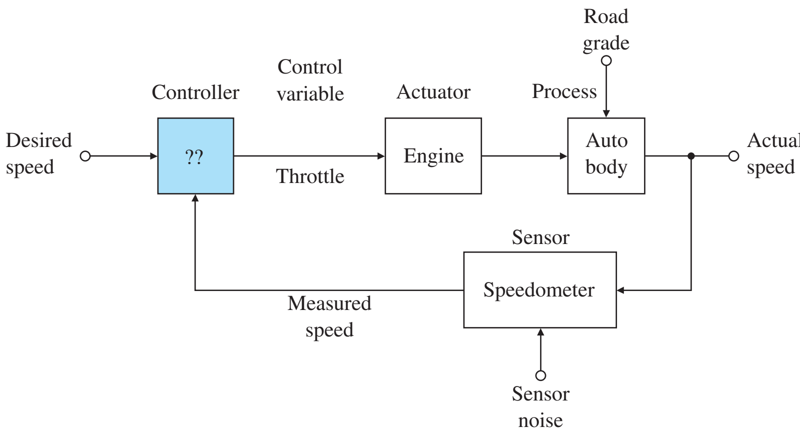

Figure 1.3
Component block diagram of automobile cruise control

need a mathematical **model** of our system in the form of a set of quantitative relationships among the variables. For this example, we ignore the dynamic response of the car and consider only the steady behavior. (Dynamics will, of course, play a major role in later chapters.) Furthermore, we assume that for the range of speeds to be used by the system, we can approximate the relations as linear. After measuring the speed of the vehicle on a level road at 65 mph, we find that a 1° change in the throttle angle (our control variable, u) causes a 10 mph change in speed (the output variable, y), hence the value 10 in the box between u and y in Fig. 1.4, which is a **block diagram** of the plant. Generally, the block diagram shows the mathematical relationships of a system in graphical form. From observations while driving up and down hills, it is found that when the grade changes by 1%, we measure a speed change of 5 mph, hence the value 0.5 in the upper box in Fig. 1.4, which reflects that a 1% grade change has half the effect of a 1° change in the throttle angle. The speedometer is found to be accurate to a fraction of 1 mph and will be considered exact. In the block diagram, the connecting lines carry signals and a block is like an ideal amplifier which multiplies the signal at its input by the value marked in the block to give the output signal. To sum two or more signals, we show lines for the signals coming into a **summer**, a circle with the summation sign Σ inside. An algebraic sign (plus or minus) beside each arrow head indicates whether the input

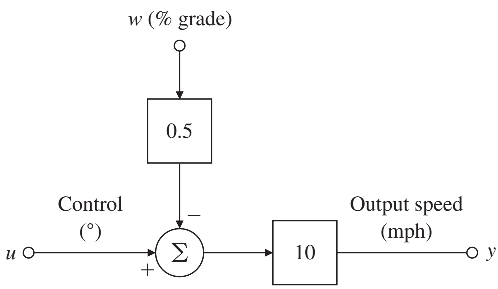

Figure 1.4
Block diagram of the cruise control plant

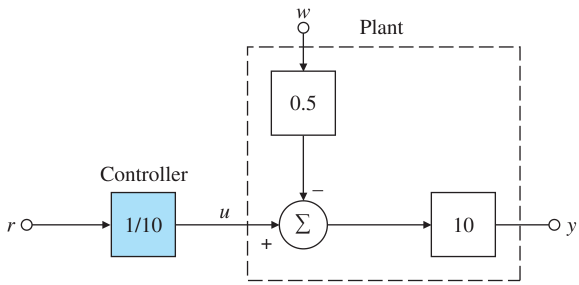

Figure 1.5
Open-loop cruise control

adds to or subtracts from the total output of the summer. For this analysis, we wish to compare the effects of a 1% grade on the output speed when the reference speed is set for 65 with and without feedback to the controller.

Open-loop control

In the first case, shown in Fig. 1.5, the controller does not use the speedometer reading but sets $u = r/10$, where r is the reference speed, which is, 65 mph. This is an example of an **open-loop control system.** The term *open-loop* refers to the fact that there is no closed path or loop around which the signals go in the block diagram; that is, the control variable u is independent of the output variable, y. In our simple example, the open-loop output speed, y_{ol}, is given by the equations

$$\begin{aligned} y_{ol} &= 10(u - 0.5w) \\ &= 10\left(\frac{r}{10} - 0.5w\right) \\ &= r - 5w. \end{aligned}$$

The error in output speed is

$$e_{ol} = r - y_{ol} \tag{1.1}$$

$$= 5w, \tag{1.2}$$

and the percent error is

$$\%\,error = 500\frac{w}{r}. \tag{1.3}$$

If $r = 65$ and the road is level, then $w = 0$ and the speed will be 65 with no error. However, if $w = 1$ corresponding to a 1% grade, then the speed will be 60 and we have a 5-mph error, which is a 7.69% error in the speed. For a grade of 2%, the speed error would be 10 mph, which is an error of 15.38%, and so on. The example shows that there would be no error when $w = 0$, but this result depends on the controller gain being the exact inverse of the plant gain of 10. In practice, the plant gain is subject to change and if it does, errors are introduced by this means also. If there is an error in the plant gain in open-loop control, the percent speed error would be the same as the percent plant-gain error.

The block diagram of a feedback scheme is shown in Fig. 1.6, where the controller gain has been set to 10. In this simple example, we have assumed that we have an ideal sensor providing a measurement of y_{cl}. In this case, the equations are

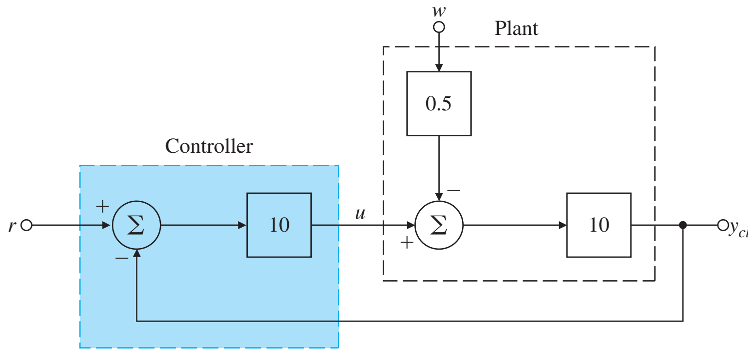

Figure 1.6
Closed-loop cruise control

$$y_{cl} = 10u - 5w,$$
$$u = 10(r - y_{cl}).$$

Combining them yields

$$y_{cl} = 100r - 100y_{cl} - 5w,$$
$$101y_{cl} = 100r - 5w,$$
$$y_{cl} = \frac{100}{101}r - \frac{5}{101}w,$$
$$e_{cl} = \frac{r}{101} + \frac{5w}{101}.$$

Thus, the feedback has reduced the sensitivity of the speed error to the grade by a factor of 101 when compared with the open-loop system. Note, however, that there is now a small speed error on level ground because even when $w = 0$,

$$y_{cl} = \frac{100}{101}r = 0.99r \text{ mph}.$$

This error will be small as long as the loop gain (product of plant and controller gains) is large.[5] If we again consider a reference speed of 65 mph and compare speeds with a 1% grade, the percent error in the output speed is

$$\% \, error = 100\frac{\frac{65 \times 100}{101} - \left(\frac{65 \times 100}{101} - \frac{5}{101}\right)}{\frac{65 \times 100}{101}} \tag{1.4}$$

$$= 100\frac{5 \times 101}{101 \times 65 \times 100} \tag{1.5}$$

$$= 0.0769\%. \tag{1.6}$$

[5] In case the error is too large, it is common practice to *reset* the reference, in this case to $\frac{101}{100}r$, so the output reaches the true desired value.

The design trade-off

The reduction of the speed sensitivity to grade disturbances and plant gain in our example is due to the loop gain of 100 in the feedback case. Unfortunately, there are limits to how high this gain can be made; when dynamics are introduced, the feedback can make the response worse than before, or even cause the system to become unstable. The dilemma is illustrated by another familiar situation where it is easy to change a feedback gain. If one tries to raise the gain of a public-address amplifier too much, the sound system will squeal in a most unpleasant way. This is a situation where the gain in the feedback loop—from the speakers to the microphone through the amplifier back to the speakers—is too much. The issue of how to get the gain as large as possible to reduce the errors without making the system become unstable is called the design trade-off and is what much of feedback control design is all about.

1.3 Feedback System Fundamentals

To achieve good control there are typical goals:

- **Stability**. The system must be stable at all times. This is an absolute requirement.
- **Tracking**. The system output must track the command reference signal as closely as possible.
- **Disturbance rejection**. The system output must be as insensitive as possible to disturbance inputs.
- **Robustness**. The aforementioned goals must be met even if the model used in the design is not completely accurate or if the dynamics of the physical system change over time.

The requirement of **stability** is basic and instability may have two causes. In the first place, the system being controlled may be unstable. This is illustrated by the Segway vehicle, which will simply fall over if the control is turned off. A second cause of instability may be the addition of feedback! Such an instability is called a "vicious circle," where the feedback signal that is circled back makes the situation worse rather than better. Stability will be discussed in much more detail in Chapters 3 and 4.

There are many examples of the requirement of having the system's output track a command signal. For example, driving a car so the vehicle stays in its lane is **command tracking**. Today, this is done by the driver; however, there are schemes now under development where the car's "autodriver" will carry out this task using feedback control while the driver does other things, for example, surfing the Internet. Similarly, flying an airplane on the approach to landing requires that a glide path be accurately tracked by the pilot or an autopilot. It is routine for today's aircraft autopilots to carry this out including the flare to the actual touchdown. The autopilot accepts inputs from the Instrument Landing System (ILS) that provides an electronic signal showing the desired landing trajectory, then commands the aircraft control surfaces so it follows the desired trajectory as closely as possible.

Disturbance rejection is one of the very oldest applications of feedback control. In this case, the "command" is simply a constant setpoint to which the output is to be held as the environment changes. A very common example of this is the room thermostat whose job it is to hold the room temperature close to the setpoint as outside temperature and wind change, and as doors and windows are opened and closed.

Finally, to design a controller for a dynamic system, it is necessary to have a **mathematical model** of the dynamic response of the system being controlled in all but the simplest cases. Unfortunately, almost all physical systems are very complex and often nonlinear. As a result, the design will usually be based on a simplified model and must be **robust** enough that the system meets its performance requirements when applied to the real device. Furthermore, as time and the environment change, even the best of models will be in error because the system dynamics have changed. Again, the design must not be too **sensitive** to these inevitable changes and it must work well enough regardless.

The **tools** available to control engineers to design and build feedback control systems have evolved over time. The development of digital computers has been especially important both as computation aids and as embedded control devices. As computation devices, computers have permitted identification of increasingly complex models and the application of very sophisticated control design methods. Also, as embedded devices, digital controllers have permitted the implementation of very complex control laws. Control engineers must not only be skilled in using these design tools, but also need to understand the concepts behind these tools to be able to make the best use of them. Also important is that the control engineer understands both the capabilities and the limitations of the controller devices available.

1.4 A Brief History

Interesting histories of early work on feedback control have been written by Mayr (1970) and Åström (2014), who trace the control of mechanisms to antiquity. Two of the earliest examples are the control of flow rate to regulate a water clock and the control of liquid level in a wine vessel, which is thereby kept full regardless of how many cups are dipped from it. The control of fluid flow rate is reduced to the control of fluid level, since a small orifice will produce constant flow if the pressure is constant, which is the case if the level of the liquid above the orifice is constant. The mechanism of the liquid-level control invented in antiquity and still used today (for example, in the water tank of the ordinary flush toilet) is the **float valve**. As the liquid level falls, so does the float, allowing the flow into the tank to increase; as the level rises, the flow is reduced and if necessary cut off. Figure 1.7 shows how a float valve operates. Notice here the sensor and actuator are not separate

Liquid-level control

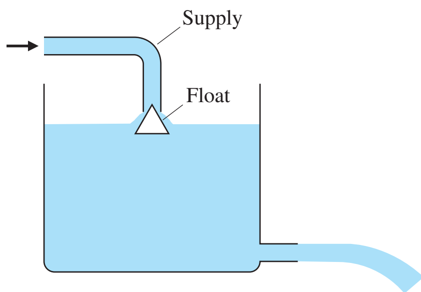

Figure 1.7
Early historical control of liquid level and flow

devices but are contained in the carefully shaped float-and-supply-tube combination.

Drebbel's incubator

A more recent invention described by Mayr (1970) is a system, designed by Cornelis Drebbel in about 1620, to control the temperature of a furnace used to heat an incubator[6] (see Fig. 1.8). The furnace consists of a box to contain the fire, with a flue at the top fitted with a damper. Inside the fire box is the double-walled incubator box, the hollow walls of which are filled with water to transfer the heat evenly to the incubator. The temperature sensor is a glass vessel filled with alcohol and mercury and placed in the water jacket around the incubator box. As the fire heats the box and water, the alcohol expands and the riser floats up, lowering the damper on the flue. If the box is too cold, the alcohol contracts, the damper is opened, and the fire burns hotter.

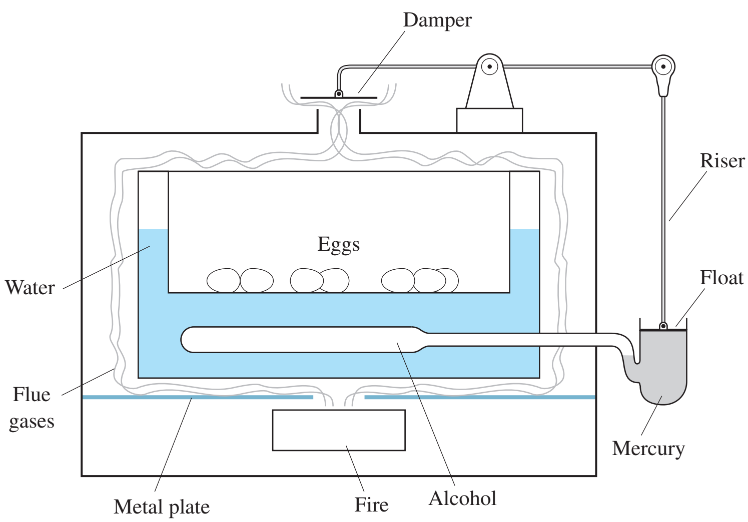

Figure 1.8
Drebbel's incubator for hatching chicken eggs

[6] French doctors introduced incubators into the care of premature babies over 100 years ago.

The desired temperature is set by the length of the riser, which sets the opening of the damper for a given expansion of the alcohol.

A famous problem in the chronicles of control systems was the search for a means to control the rotation speed of a shaft. Much early work (Fuller, 1976) seems to have been motivated by the desire to automatically control the speed of the grinding stone in a wind-driven flour mill. Of various methods attempted, the one with the most promise used a conical pendulum, or **fly-ball governor**, to measure the speed of the mill. The sails of the driving windmill were rolled up or let out with ropes and pulleys, much like a window shade, to maintain fixed speed. However, it was adaptation of these principles to the steam engine in the laboratories of James Watt around 1788 that made the fly-ball governor famous. An early version is shown in Fig. 1.9, while Figs. 1.10 and 1.11 show a close-up of a fly-ball governor and a sketch of its components.

Fly-ball governor

The action of the fly-ball governor (also called a centrifugal governor) is simple to describe. Suppose the engine is operating in equilibrium. Two weighted balls spinning around a central shaft can be seen to describe a cone of a given angle with the shaft. When a load is suddenly applied to the engine, its speed will slow, and the balls of the governor will drop to a smaller cone. Thus the ball angle is used to sense the output speed. This action, through the levers, will open the main valve to the steam chest (which is the actuator) and admit more steam to the engine, restoring most of the lost speed. To hold the steam valve at a new position, it is necessary for the fly balls to rotate at a different angle, implying that the speed under load is not exactly the same as before. We saw this effect earlier with cruise control, where feedback control gave a very small error. To recover the exact same speed in the system, it would require resetting the desired speed setting by changing the length of the rod from the lever to the valve. Subsequent inventors

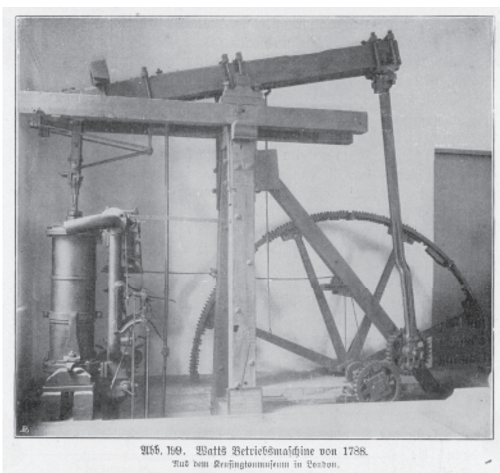

Figure 1.9
Photograph of an early Watt steam engine
Source: Chronicle/Alamy Stock Photo

Figure 1.10
Close-up of the fly-ball governor
Source: Washington Imaging/Alamy Stock Photo

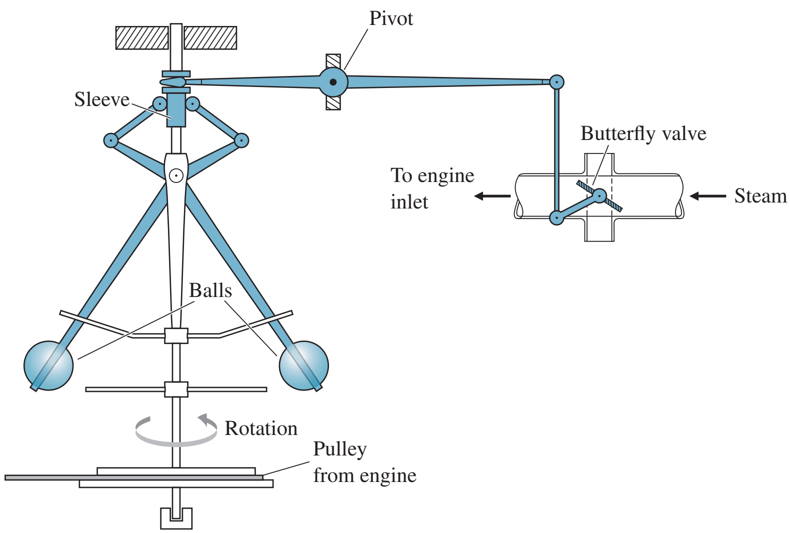

Figure 1.11
Operating parts of a fly-ball governor

introduced mechanisms that integrated the speed error to provide automatic reset. In Chapter 4, we will analyze these systems to show that such integration can result in feedback systems with zero steady-state error to constant disturbances.

Beginnings of control theory

Because Watt was a practical man, he did not engage in theoretical analysis of the governor, similar to the millwrights earlier. Fuller (1976) has traced the early development of control theory to a period of studies from Christiaan Huygens in 1673 to James Clerk Maxwell in 1868. Fuller gives particular credit to the contributions of G. B. Airy, professor of mathematics and astronomy at Cambridge University from 1826 to 1835 and Astronomer Royal at Greenwich Observatory from 1835 to 1881. Airy was concerned with speed control; if his telescopes could be rotated counter to the rotation of the earth, a fixed star could be observed for extended periods. Using the centrifugal-pendulum governor he discovered that it was capable of unstable motion—"and the machine (if I may so express myself) became perfectly wild" (Airy, 1840; quoted in Fuller, 1976). According to Fuller, Airy was the first worker to discuss instability in a feedback control system and the first to analyze such a system using differential equations. These attributes signal the beginnings of the study of feedback control dynamics.

Stability analysis

The first systematic study of the stability of feedback control was apparently given in the paper "On Governors" by Maxwell (1868).[7] In this paper, Maxwell developed the differential equations of the governor, linearized them about equilibrium, and stated that stability depends on the roots of a certain (characteristic) equation having negative real parts. Maxwell attempted to derive conditions on the coefficients of a polynomial that would hold if all the roots had negative real parts. He was successful only for second- and third-order cases. Determining criteria for stability was the problem for the Adams Prize of 1877, which was won by E. J. Routh.[8] His criterion, developed in his essay, remains of sufficient interest that control engineers are still learning how to apply his simple technique. Analysis of the characteristic equation remained the foundation of control theory until the invention of the electronic feedback amplifier by H. S. Black in 1927 at Bell Telephone Laboratories.

Shortly after publication of Routh's work, the Russian mathematician Lyapunov (1892) began studying the question of stability of motion. His studies were based on the nonlinear differential equations of motion, and also included results for linear equations that are equivalent to Routh's criterion. His work was fundamental to what is now called the state-variable approach to control theory, but was not introduced into the control literature until about 1958.

Frequency response

The development of the feedback amplifier is briefly described in an interesting article based on a talk by Bode (1960) reproduced in Bellman and Kalaba (1964). With the introduction of electronic amplifiers, long-distance telephoning became possible in the decades following World War I. However, as distances increased, so did the loss of electrical energy; in spite of using larger-diameter wires, increasing numbers of amplifiers were needed to replace the lost energy. Unfortunately, large numbers of amplifiers resulted in much distortion since the small nonlinearity of the vacuum tubes then used in electronic amplifiers were

[7] An exposition of Maxwell's contribution is given in Fuller (1976).

[8] E. J. Routh was first academically in his class at Cambridge University in 1854, while J. C. Maxwell was second. In 1877, Maxwell was on the Adams Prize Committee that chose the problem of stability as the topic for the year.

multiplied many times. To solve the problem of reducing distortion, Black proposed the feedback amplifier. As mentioned earlier in connection with the automobile cruise control, the more we wish to reduce errors (or distortion), the more feedback we need to apply. The loop gain from actuator to plant to sensor to actuator must be made very large. With high gain the feedback loop begins to squeal and is unstable. Here was Maxwell's and Routh's stability problem again, except that in this technology, the dynamics were so complex (with differential equations of order 50 being common) that Routh's criterion was not very helpful. So the communications engineers at Bell Telephone Laboratories, familiar with the concept of frequency response and the mathematics of complex variables, turned to complex analysis. In 1932, H. Nyquist published a paper describing how to determine stability from a graphical plot of the loop frequency response. From this theory developed an extensive methodology of feedback-amplifier design described by Bode (1945) and still extensively used in the design of feedback controls. Nyquist and Bode plots will be discussed in more detail in Chapter 6.

Simultaneous with the development of the feedback amplifier, feedback control of industrial processes was becoming standard. This field, characterized by processes that are not only highly complex but also nonlinear and subject to relatively long time delays between actuator and sensor, developed the concept of **proportional-integral-derivative (PID) control**. The PID controller was first described by Callender et al. (1936). This technology was based on extensive experimental work and simple linearized approximations to the system dynamics. It led to standard experiments suitable to application in the field and eventually to satisfactory "tuning" of the coefficients of the PID controller. (PID controllers will be covered in Chapter 4.) Also under development at this time were devices for guiding and controlling aircraft; especially important was the development of sensors for measuring aircraft altitude and speed. An interesting account of this branch of control theory is given in McRuer (1973).

PID control

An enormous impulse was given to the field of feedback control during World War II. In the United States, engineers and mathematicians at the MIT Radiation Laboratory combined their knowledge to bring together not only Bode's feedback amplifier theory and the PID control of processes, but also the theory of stochastic processes developed by Wiener (1930). The result was the development of a comprehensive set of techniques for the design of **servomechanisms**, as control mechanisms came to be called. Much of this work was collected and published in the records of the Radiation Laboratory by James et al. (1947).

Another approach to control systems design was introduced in 1948 by W. R. Evans, who was working in the field of guidance and control of aircraft. Many of his problems involved unstable or neutrally stable dynamics, which made the frequency methods difficult, so he suggested returning to the study of the characteristic equation that had been the basis of the work of Maxwell and Routh nearly 70 years earlier. However, Evans developed techniques and rules allowing one to follow graphically the paths of the roots of the characteristic equation as a parameter was changed. His method, the **root locus**, is suitable for design as well as for stability analysis and remains an important technique today. The root-locus method developed by Evans will be covered in Chapter 5.

Root locus

During the 1950s, several authors, including R. Bellman and R. E. Kalman in the United States and L. S. Pontryagin in the U.S.S.R., began again to consider the ordinary differential equation (ODE) as a model for control systems. Much of this work was stimulated by the new field of control of artificial earth satellites, in which the ODE is a natural form for writing the model. Supporting this endeavor were digital computers, which could be used to carry out calculations unthinkable 10 years before. (Now, of course, these calculations can be done by any engineering student with a laptop computer.) The work of Lyapunov was translated into the language of control at about this time, and the study of optimal controls, begun by Wiener and Phillips during World War II, was extended to optimizing trajectories of nonlinear systems based on the calculus of variations. Much of this work was presented at the first conference of the newly formed International Federation of Automatic Control held in Moscow in 1960.[9] This work did not use the frequency response or the characteristic equation but worked directly with the ODE in "normal" or "state" form and typically called for extensive use of computers. Even though the foundations of the study of ODEs were laid in the late 19th century, this approach is now often called **modern control** to distinguish it from **classical control**, which uses Laplace transforms and complex variable methods of Bode and others. In the period from the 1970s continuing through the present, we find a growing body of work that seeks to use the best features of each technique.

State-variable design

Thus, we come to the current state of affairs where the principles of control are applied in a wide range of disciplines, including every branch of engineering. The well-prepared control engineer needs to understand the basic mathematical theory that underlies the field and must be able to select the best design technique suited to the problem at hand. With the ubiquitous use of computers, it is especially important that the engineer is able to use his or her knowledge to guide and verify calculations done on the computer.[10]

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[9]Optimal control gained a large boost when Bryson and Denham (1962) showed that the path of a supersonic aircraft should actually dive at one point in order to reach a given altitude in minimum time. This nonintuitive result was later demonstrated to skeptical fighter pilots in flight tests.

[10]For more background on the history of control, see the survey papers appearing in the *IEEE Control Systems Magazine* of November 1984 and June 1996.

1.5 An Overview of the Book

The central purpose of this book is to introduce the most important techniques for single-input–single-output control systems design. **Chapter 2** will review the techniques necessary to obtain physical models of the dynamic systems that we wish to control. These include model making for mechanical, electric, electromechanical, and a few other physical systems, including a simple model for a quadrotor drone, which will be used in subsequent chapters. Chapter 2 will also briefly describe the linearization of nonlinear models, although this will be discussed more thoroughly in **Chapter 9**.

In **Chapter 3** and **Appendix A**, we will discuss the analysis of dynamic response using Laplace transforms along with the relationship between time response and the poles and zeros of a transfer function. The chapter also includes a discussion of the critical issue of system stability, including the Routh test.

In **Chapter 4**, we will cover the basic equations and features of feedback. An analysis of the effects of feedback on disturbance rejection, tracking accuracy, sensitivity to parameter changes, and dynamic response will be given. The idea of elementary PID control is discussed.

In **Chapters 5, 6**, and **7**, we introduce the techniques for realizing the control objectives first identified in Chapter 4 in more complex dynamic systems. These include the root locus, frequency response, and state variable techniques. These are alternative means to the same end and have different advantages and disadvantages as guides to design of controls. The methods are fundamentally complementary, and each needs to be understood to achieve the most effective control systems design.

In **Chapter 8**, we will develop the ideas of implementing controllers in a digital computer. The chapter addresses how one "digitizes" the control equations developed in Chapters 4 through 7, how the sampling introduces a delay that tends to destabilize the system, and how the sample rate needs to be a certain multiple of the system frequencies for good performance. Just as the Laplace transform does for nonsampled signals, the analysis of sampled systems requires another analysis tool—the z-transform—and that tool is described and its use is illustrated.

Most real systems are nonlinear to some extent. However, the analyses and design methods in most of the book up to here are for linear systems. In **Chapter 9**, we will explain why the study of linear systems is pertinent, why it is useful for design even though most systems are nonlinear, and how designs for linear systems can be modified to handle many common nonlinearities in the systems being controlled. The chapter will cover saturation, describing functions, adaptive control and the anti-windup controller, and contains a brief introduction to Lyapunov stability theory.

Application of all the techniques to problems of substantial complexity will be discussed in **Chapter 10**. The design methods discussed in Chapters 4–7 are all brought to bear simultaneously on specific case studies which are representative of real world problems. These cases are somewhat simplified versions of control systems that are in use today in satellites on orbit, in most commercial aircraft, in all automobiles sold in the Western world today, in semiconductor manufacturing throughout the world, and in the drones being used in many fields.

Computer aids

Control designers today make extensive use of computer-aided control systems design software that is commercially available. Furthermore, most instructional programs in control systems design make software tools available to the students. The most widely used software for the purpose are Matlab® and Simulink® from The MathWorks. Matlab routines have been included throughout the text to help illustrate this method of solution and many problems require computer aids for solution. Many of the figures in the book were created using Matlab and the files for their creation are available free of charge on the web at **www.pearsonglobaleditions.com**. Students and instructors are invited to use these files as it is believed that they should be helpful in learning how to use computer methods to solve control problems.

Needless to say, many topics are not treated in the book. We do not extend the methods to multivariable controls, which are systems with more than one input and/or output, except as part of the case study of the rapid thermal process in Chapter 10. Nor is optimal control treated in more than a very introductory manner in Chapter 7.

Also beyond the scope of this text is a detailed treatment of the experimental testing and modeling of real hardware, which is the ultimate test of whether any design really works. The book concentrates on analysis and design of linear controllers for linear plant models—not because we think that is the final test of a design, but because that is the best way to grasp the basic ideas of feedback and is usually the first step in arriving at a satisfactory design. We believe that mastery of the material here will provide a foundation of understanding on which to build knowledge of the actual physical behavior of control systems—a foundation strong enough to allow one to build a personal design method in the tradition of all those who worked to give us the knowledge we present here.

SUMMARY

- **Control** is the process of making a system variable adhere to a particular value, called the **reference value**. A system designed to follow a changing reference is called **tracking control** or a **servo**. A system designed to maintain an output fixed regardless of the disturbances present is called a **regulating control** or a **regulator**.
- Two kinds of control were defined and illustrated based on the information used in control and named by the resulting structure. In **open-loop control**, the system does *not* measure the output and there is no correction of the actuating signal to make that output conform to the reference signal. In **closed-loop control**, the system

includes a sensor to measure the output and uses **feedback** of the sensed value to influence the control variable.

- A simple feedback system consists of the **process** (or **plant**) whose output is to be controlled, the **actuator** whose output causes the process output to change, a **reference** command signal, and **output sensors** that measure these signals, and the **controller** that implements the logic by which the control signal that commands the actuator is calculated.
- **Block diagrams** are helpful for visualizing system structure and the flow of information in control systems. The most common block diagrams represent the mathematical relationships among the signals in a control system.
- A well-designed feedback control system will be **stable**, **track a desired input** or setpoint, **reject disturbances**, and be insensitive (or **robust**) to changes in the **math model** used for design.
- The theory and design techniques of control have come to be divided into two categories: **classical control** methods use Laplace transforms (or z-transform) and were the dominant methods for control design until **modern control** methods based on ODEs in state form were introduced into the field starting in the 1960s. Many connections have been discovered between the two categories and well-prepared engineers must be familiar with both techniques.

REVIEW QUESTIONS

1.1 What are the main components of a feedback control system?

1.2 What is the purpose of the sensor?

1.3 Give three important properties of a good sensor.

1.4 What is the purpose of the actuator?

1.5 Give three important properties of a good actuator.

1.6 What is the purpose of the controller? Give the input(s) and output(s) of the controller.

1.7 What physical variable is measured by a tachometer?

1.8 Describe three different techniques for measuring temperature.

1.9 Why do most sensors have an electrical output, regardless of the physical nature of the variable being measured?

PROBLEMS

1.1 Draw a component block diagram for each of the following feedback control systems:

(a) The manual steering system of an automobile

(b) Drebbel's incubator

(c) The water level controlled by a float and valve

(d) Watt's steam engine with fly-ball governor

In each case, indicate the location of the elements listed below and give the units associated with each signal:

- The process
- The process desired output signal
- The sensor
- The actuator
- The actuator output signal
- The controller
- The controller output signal
- The reference signal
- The error signal

Notice that in a number of cases the same physical device may perform more than one of these functions.

1.2 Identify the physical principles and describe the operation of the thermostat in your home or office.

1.3 A machine for making paper is diagrammed in Fig. 1.12. There are two main parameters under feedback control: the density of fibers as controlled by the consistency of the thick stock that flows from the headbox onto the wire, and the moisture content of the final product that comes out of the dryers. Stock from the machine chest is diluted by white water returning from under the wire as controlled by a control valve (CV). A meter supplies a reading of the consistency. At the "dry end" of the machine, there is a moisture sensor. Draw a block diagram and identify the nine components listed in Problem 1.1 part (d) for the following:

(a) Control of consistency

(b) Control of moisture

1.4 Many variables in the human body are under feedback control. For each of the following controlled variables, draw a block diagram showing the process being controlled, the sensor that measures the variable, the actuator that causes it to increase and/or decrease, the information path that completes the feedback path, and the disturbances that upset the variable. You may need to consult an encyclopedia or textbook on human physiology for information on this problem.

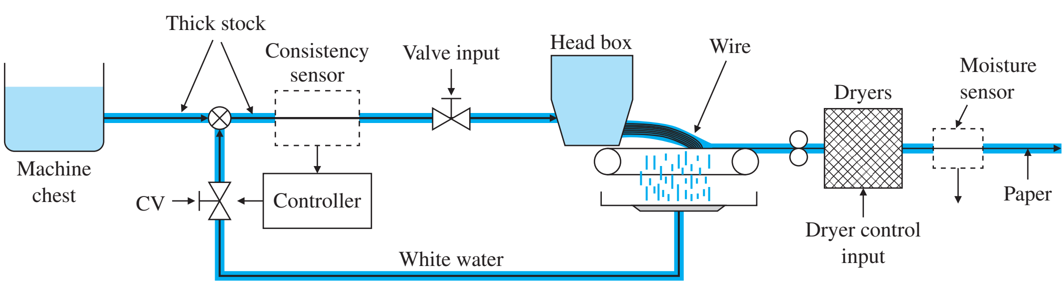


Figure 1.12

A papermaking machine

(a) Blood pressure
(b) Blood sugar concentration
(c) Heart rate
(d) Eye-pointing angle
(e) Eye-pupil diameter

1.5 Draw a block diagram of the components for an elevator-position control. Indicate how you would measure the position of the elevator car. Consider a combined coarse and fine measurement system. What accuracies do you suggest for each sensor? Your system should be able to correct for the fact that in elevators for tall buildings there is significant cable stretch as a function of cab load.

1.6 Feedback control requires being able to sense the variable being controlled. Because electrical signals can be transmitted, amplified, and processed easily, often we want to have a sensor whose output is a voltage or current proportional to the variable being measured. Describe a sensor that would give an electrical output proportional to the following:

(a) Temperature
(b) Pressure
(c) Liquid level
(d) Flow of liquid along a pipe (or blood along an artery)
(e) Linear position
(f) Rotational position
(g) Linear velocity
(h) Rotational speed
(i) Translational acceleration
(j) Torque

1.7 Each of the variables listed in Problem 1.6 can be brought under feedback control. Describe an actuator that could accept an electrical input and be used to control the variables listed. Give the units of the actuator output signal.

1.8 *Feedback in Biology*

(a) *Negative Feedback in Biology*: When a person is under long-term stress (say, a couple of weeks before an exam!), hypothalamus (in the brain) secretes a hormone called Corticotropin Releasing Factor (CRF) which binds to a receptor in the pituitary gland stimulating it to produce Adrenocorticotropic hormone (ACTH), which in turn stimulates the adrenal cortex (outer part of the adrenal glands) to release the stress hormone Glucocorticoid (GC). This in turn shuts down (turns off the stress response) for both CRF and ACTH production by negative feedback via the bloodstream until GC returns to its normal level. Draw a block diagram of this closed-loop system.

(b) *Positive Feedback in Biology*: This happens in some unique circumstances. Consider the birth process of a baby. Pressure from the head of the baby going through the birth canal causes contractions via secretion of a hormone called oxytocin which causes more pressure which in turn intensifies contractions. Once the baby is born, the system goes back to normal (negative feedback). Draw a block diagram of this closed-loop system.

2

Dynamic Models

A Perspective on Dynamic Models

The overall goal of feedback control is to use feedback to cause the output variable of a dynamic process to follow a desired reference variable accurately, regardless of the reference variable's path and regardless of any external disturbances or any changes in the dynamics of the process. This complex design goal is met by a number of simple, distinct steps. The first of these is to develop a mathematical description (called a **dynamic model** or **mathematical model**) of the process to be controlled. The term **model**, as it is used and understood by control engineers, means a set of differential equations that describe the dynamic behavior of the process. A model can be obtained using principles of the underlying physics or by testing a prototype of the device, measuring its response to inputs, and using the data to construct an analytical model. We will focus only on using physics in this chapter. There are entire books written on experimentally determining models, sometimes called **system identification**, and these techniques will be described very briefly in Chapter 3. A careful control system designer will typically rely on at least some experiments to verify the accuracy of the model when it is derived from physical principles.

In many cases, the modeling of complex processes is difficult and expensive, especially when the important steps of building and testing prototypes are included. However, in this introductory text, we will focus on the most basic principles of modeling for the most common physical systems. More comprehensive sources and specialized texts will be referenced throughout where appropriate for those wishing more detail.

Source: burnell1/123RF

In later chapters, we will explore a variety of analysis methods for dealing with the dynamic equations and their solution for purposes of designing feedback control systems.

Chapter Overview

The fundamental step in building a dynamic model is writing the dynamic equations for the system. Through discussion and a variety of examples, Section 2.1 demonstrates how to write the dynamic equations for a variety of mechanical systems. In addition, the section demonstrates the use of Matlab to find the time response of a simple system to a step input. Furthermore, the ideas of transfer functions and block diagrams are introduced, along with the idea that problems can also be solved via Simulink.

Electric circuits and electromechanical systems will be modeled in Sections 2.2 and 2.3, respectively.

For those wanting modeling examples for more diverse dynamic systems, Section 2.4, which is optional, will extend the discussion to heat- and fluid-flow systems.

The chapter then concludes with Section 2.5, a discussion of the history behind the discoveries that led to the knowledge that we take for granted today.

The differential equations developed in modeling are often nonlinear. Because nonlinear systems are significantly more challenging to solve than linear ones, and because linear models are usually adequate for purposes of control design, the emphasis in the early chapters is primarily on linear systems. However, we do show how to linearize simple nonlinearities in this chapter and show how to use Simulink to numerically solve for the motion of a nonlinear system. A much more extensive discussion of linearization and analysis of nonlinear systems is contained in Chapter 9.

In order to focus on the important first step of developing mathematical models, we will defer explanation of the computational methods used to solve the dynamic equations developed in this chapter until Chapter 3.

2.1 Dynamics of Mechanical Systems

2.1.1 Translational Motion

Newton's law for translational motion

The cornerstone for obtaining a mathematical model, or the **dynamic equations**,[1] for any mechanical system is Newton's law,

$$\mathbf{F} = m\mathbf{a}, \tag{2.1}$$

[1] For systems with moving parts, these equations are typically referred to as the **"equations of motion."**

where

$\mathbf{F}$ = the vector sum of all forces applied to each body in a system, newtons (N),

$\mathbf{a}$ = the vector acceleration of each body with respect to an **inertial reference frame** (that is, one that is neither accelerating nor rotating with respect to the stars); often called **inertial acceleration**, m/sec^2,

m = mass of the body, kg.

Note that here in Eq. (2.1), as throughout the text, we use the convention of boldfacing the type to indicate that the quantity is a matrix or vector, possibly a vector function.

A force of 1 N will impart an acceleration of 1 m/sec^2 to a mass of 1 kg. The "weight" of an object is mg, where g is the acceleration of gravity (= 9.81 m/sec^2), which is the quantity measured on scales. Scales are typically calibrated in kilograms, which is used as a direct measure of mass assuming the standard value for g.

Use of free-body diagram in applying Newton's law

Application of this law typically involves defining convenient coordinates to account for the body's motion (position, velocity, and acceleration), determining the forces on the body using a free-body diagram, then writing the equations of motion from Eq. (2.1). The procedure is simplest when the coordinates chosen express the position with respect to an inertial reference frame because, in this case, the accelerations needed for Newton's law are simply the second derivatives of the position coordinates.

EXAMPLE 2.1 *A Simple System; Cruise Control Model*

1. Write the equations of motion for the speed and forward motion of the car shown in Fig. 2.1, assuming the engine imparts a force u as shown. Take the Laplace transform of the resulting differential equation and find the transfer function between the input u and the output v.

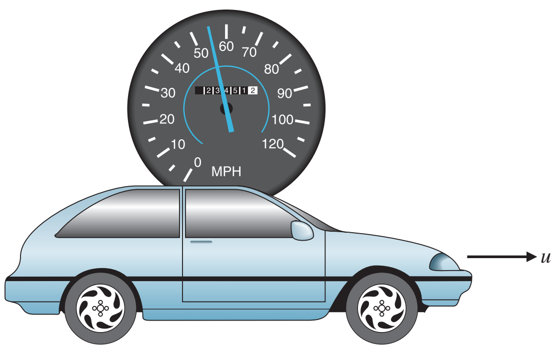


Figure 2.1
Cruise control model

2. Use Matlab to find the response of the velocity of the car for the case in which the input jumps from being $u = 0$ at time $t = 0$ to a constant $u = 500$ N thereafter. Assume the car mass m is 1000 kg and viscous drag coefficient, $b = 50$ N·sec/m.

Solution

1. **Equations of motion:** For simplicity, we assume the rotational inertia of the wheels is negligible, and that there is friction retarding the motion of the car that is proportional to the car's speed with a proportionality constant, b.[2] The car can then be approximated for modeling purposes using the free-body diagram seen in Fig. 2.2, which defines coordinates, shows all forces acting on the body (heavy lines), and indicates the acceleration (dashed line). The coordinate of the car's position, x, is the distance from the reference line shown and is chosen so positive is to the right. Note in this case, the inertial acceleration is simply the second derivative of x (that is, $\mathbf{a} = \ddot{x}$) because the car position is measured with respect to an inertial reference frame. The equation of motion is found using Eq. (2.1). The friction force acts opposite to the direction of motion; therefore it is drawn opposite to the direction of positive motion and entered as a negative force in Eq. (2.1). The result is

$$u - b\dot{x} = m\ddot{x}, \tag{2.2}$$

or

$$\ddot{x} + \frac{b}{m}\dot{x} = \frac{u}{m}. \tag{2.3}$$

For the case of the automotive cruise control where the variable of interest is the speed, v ($=\dot{x}$), the equation of motion becomes

$$\dot{v} + \frac{b}{m}v = \frac{u}{m}. \tag{2.4}$$

The solution of such an equation will be covered in detail in Chapter 3; however, the essence is that you assume a solution of

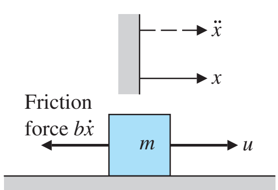


Figure 2.2
Free-body diagram for cruise control

[2] If the speed is v, the aerodynamic portion of the friction force is actually proportional to v^2. We have assumed it to be linear here for simplicity.

the form $v = V_o e^{st}$ given an input of the form $u = U_o e^{st}$. Then, since $\dot{v} = sV_o e^{st}$, the differential equation can be written as[3]

$$\left(s + \frac{b}{m}\right) V_o e^{st} = \frac{1}{m} U_o e^{st}. \tag{2.5}$$

The e^{st} term cancels out, and we find that

$$\frac{V_o}{U_o} = \frac{\frac{1}{m}}{s + \frac{b}{m}}. \tag{2.6}$$

For reasons that will become clear in Chapter 3, this is often written using capital letters to signify that it is the "transform" of the solution, or

$$\frac{V(s)}{U(s)} = \frac{\frac{1}{m}}{s + \frac{b}{m}}. \tag{2.7}$$

Transfer function

This expression of the differential equation (2.4) is called the **transfer function** and will be used extensively in later chapters. Note that, in essence, we have substituted s for d/dt in Eq. (2.4). This transfer funtion serves as a math model that relates the car's velocity to the forces propelling the car, that is, inputs from the accelerator pedal. Transfer functions of a system will be used in later chapters to design feedback controllers such as a cruise control device found in many modern cars.

2. **Time response:** The dynamics of a system can be prescribed to Matlab in terms of its transfer function as can be seen in the Matlab statements below that implements Eq. (2.7). The step function in Matlab calculates the time response of a linear system to a unit step input. Because the system is linear, the output for this case can be multiplied by the magnitude of the input step to derive a step response of any amplitude. Equivalently, sys can be multiplied by the magnitude of the input step.

Step response with Matlab

The statements

```
s=tf('s');                     % sets up the mode to define the
                                 transfer function
sys = (1/1000)/(s + 50/1000);  % defines the transfer function from
                                 Eq. (2.7) with the numbers filled in.
step(500*sys);                 % plots the step response for u = 500.
```

calculate and plot the time response of velocity for an input step with a 500-N magnitude. The step response is shown in Fig. 2.3.

Newton's law also can be applied to systems with more than one mass. In this case, it is particularly important to draw the free-body

[3] The use of an operator for differentiation was developed by Cauchy in about 1820 based on the Laplace transform, which was developed in about 1780. In Chapter 3, we will show how to derive transfer functions using the Laplace transform (refer to Gardner and Barnes, 1942).

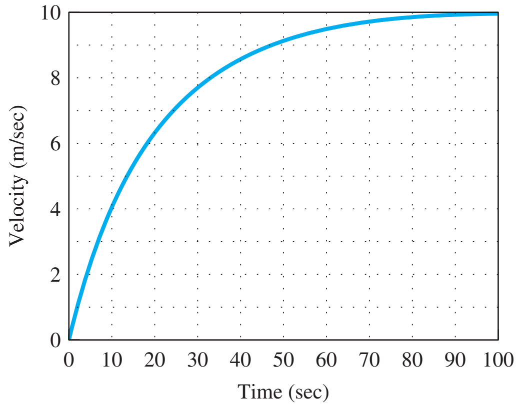


Figure 2.3
Response of the car velocity to a step in u

diagram of each mass, showing the applied external forces as well as the equal and opposite internal forces that act from each mass on the other.

EXAMPLE 2.2 *A Two-Mass System: Suspension Model*

Figure 2.4 shows an automobile suspension system. Write the equations of motion for the automobile and wheel motion assuming one-dimensional vertical motion of one quarter of the car mass above one wheel. A system consisting of one of the four-wheel suspensions is usually referred to as a **quarter-car model**. The system can be approximated by the simplified system shown in Fig. 2.5 where two spring constants and a damping coefficient are defined. Assume the model is for a car with a mass of 1580 kg, including the four wheels, which have a mass of 20 kg each. By placing a known weight (an author) directly over a wheel and measuring the car's deflection, we find that $k_s = 130{,}000$ N/m. Measuring the wheel's deflection for the same applied weight, we find that $k_w \simeq 1{,}000{,}000$ N/m. By using the step response data in Fig. 3.19(b) and qualitatively observing that the car's response to a step change matches the damping coefficient curve for $\zeta = 0.7$ in the figure, we conclude that $b = 9800$ N·sec/m.

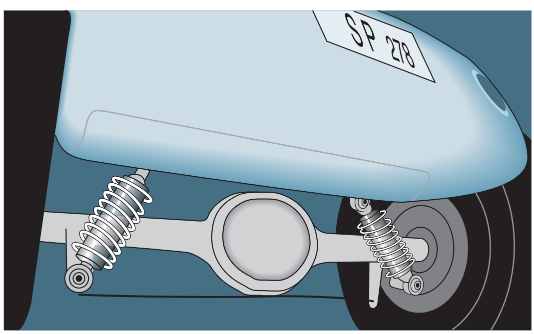


Figure 2.4
Automobile suspension

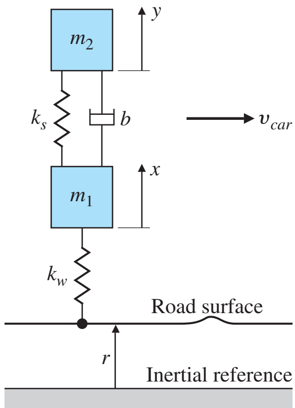


Figure 2.5
The quarter-car model

Solution. The system can be approximated by the simplified system shown in Fig. 2.5. The coordinates of the two masses, x and y, with the reference directions as shown, are the displacements of the masses from their equilibrium conditions. The equilibrium positions are offset from the springs' unstretched positions because of the force of gravity. The shock absorber is represented in the schematic diagram by a dashpot symbol with friction constant b. The magnitude of the force from the shock absorber is assumed to be proportional to the rate of change of the relative displacement of the two masses—that is, the force $= b(\dot{y} - \dot{x})$. The force of gravity could be included in the free-body diagram; however, its effect is to produce a constant offset of x and y. By defining x and y to be the distance from the equilibrium position, the need to include the gravity forces is eliminated.

The force from the car suspension acts on both masses in proportion to their relative displacement with spring constant k_s. Figure 2.6 shows the free-body diagram of each mass. Note the forces from the spring on the two masses are equal in magnitude but act in opposite directions, which is also the case for the damper. A positive displacement y of mass m_2 will result in a force from the spring on m_2 in the direction shown and a force from the spring on m_1 in the direction shown. However, a positive displacement x of mass m_1 will result in a force from the spring k_s on m_1 in the opposite direction to that drawn in Fig. 2.6, as indicated by the *minus* x term for the spring force.

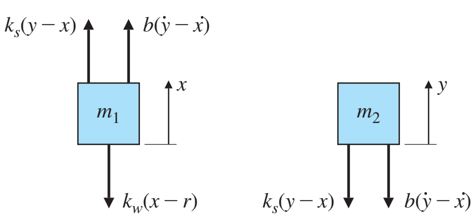


Figure 2.6
Free-body diagrams for suspension system

The lower spring k_w represents the tire compressibility, for which there is insufficient damping (velocity-dependent force) to warrant including a dashpot in the model. The force from this spring is proportional to the distance the tire is compressed and the nominal equilibrium force would be that required to support m_1 and m_2 against gravity. By defining x to be the distance from equilibrium, a force will result if either the road surface has a bump (r changes from its equilibrium value of zero) or the wheel bounces (x changes). The motion of the simplified car over a bumpy road will result in a value of $r(t)$ that is not constant.

As previously noted, there is a constant force of gravity acting on each mass; however, this force has been omitted, as have been the equal and opposite forces from the springs. Gravitational forces can always be omitted from vertical-spring mass systems (1) if the position coordinates are defined from the equilibrium position that results when gravity is acting, and (2) if the spring forces used in the analysis are actually the perturbation in spring forces from those forces acting at equilibrium.

Applying Eq. (2.1) to each mass, and noting that some forces on each mass are in the negative (down) direction, yields the system of equations

$$b(\dot{y} - \dot{x}) + k_s(y - x) - k_w(x - r) = m_1\ddot{x},$$
$$-k_s(y - x) - b(\dot{y} - \dot{x}) = m_2\ddot{y}.$$

Some rearranging results in

$$\ddot{x} + \frac{b}{m_1}(\dot{x} - \dot{y}) + \frac{k_s}{m_1}(x - y) + \frac{k_w}{m_1}x = \frac{k_w}{m_1}r, \tag{2.8}$$

$$\ddot{y} + \frac{b}{m_2}(\dot{y} - \dot{x}) + \frac{k_s}{m_2}(y - x) = 0. \tag{2.9}$$

Check for sign errors

The most common source of error in writing equations for systems such as these are sign errors. The method for keeping the signs straight in the preceding development entailed mentally picturing the displacement of the masses and drawing the resulting force in the direction that the displacement would produce. Once you have obtained the equations for a system, a check on the signs for systems that are obviously stable from physical reasoning can be quickly carried out. As we will see when we study stability in Section 3.6 of Chapter 3, a stable system always has the same signs on similar variables. For this system, Eq. (2.8) shows that the signs on the $\ddot{x}$, $\dot{x}$, and x terms are all positive, as they must be for stability. Likewise, the signs on the $\ddot{y}$, $\dot{y}$, and y terms are all positive in Eq. (2.9).

The transfer function is obtained in a similar manner as before for zero initial conditions. Substituting s for d/dt in the differential equations yields

$$s^2X(s) + s\frac{b}{m_1}(X(s) - Y(s)) + \frac{k_s}{m_1}(X(s) - Y(s)) + \frac{k_w}{m_1}X(s) = \frac{k_w}{m_1}R(s),$$

$$s^2Y(s) + s\frac{b}{m_2}(Y(s) - X(s)) + \frac{k_s}{m_2}(Y(s) - X(s)) = 0,$$

which can also be written in matrix form as

$$\begin{bmatrix} s^2 + s\frac{b}{m_1} + \frac{k_s}{m_1} + \frac{k_w}{m_1} & -s\frac{b}{m_1} - \frac{k_s}{m_1} \\ -s\frac{b}{m_2} - \frac{k_s}{m_2} & s^2 + s\frac{b}{m_2} + \frac{k_s}{m_2} \end{bmatrix} \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} + \begin{bmatrix} \frac{k_w}{m_1} \\ 0 \end{bmatrix} R(s).$$

for which Cramer's Rule (see Appendix WB) can be used to find the transfer function

$$\frac{Y(s)}{R(s)} = \frac{\frac{k_w b}{m_1 m_2}\left(s + \frac{k_s}{b}\right)}{s^4 + \left(\frac{b}{m_1} + \frac{b}{m_2}\right)s^3 + \left(\frac{k_s}{m_1} + \frac{k_s}{m_2} + \frac{k_w}{m_1}\right)s^2 + \left(\frac{k_w b}{m_1 m_2}\right)s + \frac{k_w k_s}{m_1 m_2}}.$$

To determine numerical values, we subtract the mass of the four wheels from the total car mass of 1580 kg and divide it by 4 to find that $m_2 = 375$ kg. The wheel mass was measured directly to be $m_1 = 20$ kg. Therefore, the transfer function with the numerical values is

$$\frac{Y(s)}{R(s)} = \frac{1.31e06(s + 13.3)}{s^4 + (516.1)s^3 + (5.685e04)s^2 + (1.307e06)s + 1.733e07}.$$

We will see in Chapter 3 (and later chapters) how this sort of transfer function will allow us to find the response of the car body to inputs resulting from the car motion over a bumpy road.

2.1.2 Rotational Motion

Newton's law for rotational motion

Application of Newton's law to one-dimensional rotational systems requires that Eq. (2.1) be modified to

$$M = I\alpha, \tag{2.10}$$

where

M = the sum of all external moments about the center of mass of a body, N · m,

I = the body's mass moment of inertia about its center of mass, kg·m^2,

α = the angular acceleration of the body, rad/sec^2.

EXAMPLE 2.3 *Rotational Motion: Satellite Attitude Control Model*

Satellites, as shown in Fig. 2.7, usually require attitude control so antennas, sensors, and solar panels are properly oriented. Antennas are usually pointed toward a particular location on earth, while solar panels need to be oriented toward the sun for maximum power generation. To

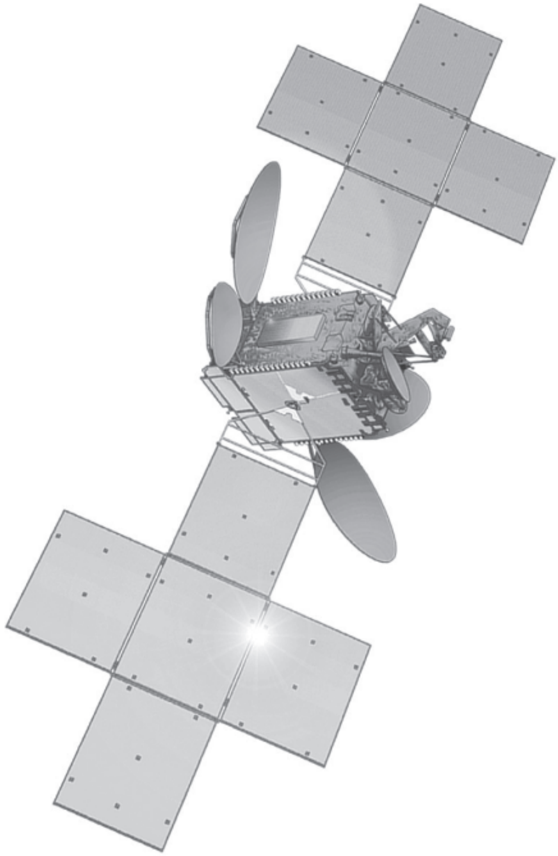


Figure 2.7
Communications satellite
Source: Courtesy Thaicom PLC and Space Systems/Loral

gain insight into the full three-axis attitude control system, it is helpful to consider one axis at a time. Write the equations of motion for one axis of this system then show how they would be depicted in a block diagram. In addition, determine the transfer function of this system and construct the system as if it were to be evaluated via Matlab's Simulink.

Solution. Figure 2.8 depicts this case, where motion is allowed only about the axis perpendicular to the page. The angle θ that describes the satellite orientation must be measured with respect to an inertial reference—that is, a reference that has no angular acceleration. The control force comes from reaction jets that produce a moment of $F_c d$ about the mass center. There may also be small disturbance moments M_D on the satellite, which arise primarily from solar pressure acting on any asymmetry in the solar panels. Applying Eq. (2.10) yields the equation of motion

$$F_c d + M_D = I\ddot{\theta}. \tag{2.11}$$

Double-integrator plant

The output of this system, θ, results from integrating the sum of the input torques twice; hence, this type of system is often referred to as

Figure 2.8
Satellite control schematic

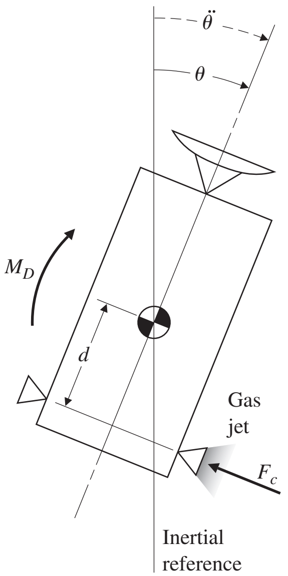


Figure 2.9
Block diagrams representing Eq. (2.11) in the upper half and Eq. (2.12) in the lower half

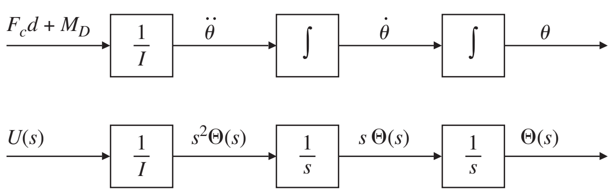
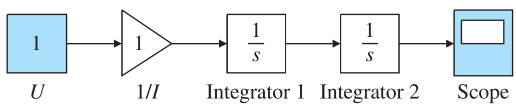


Figure 2.10
Simulink block diagram of the double-integrator plant



the **double-integrator plant**. The transfer function can be obtained as described for Eq. (2.7) and is

$$\frac{\Theta(s)}{U(s)} = \frac{1}{I}\frac{1}{s^2}, \tag{2.12}$$

where $U = F_c d + M_D$. In this form, the system is often referred to as the $1/s^2$ **plant**.

$1/s^2$ plant

Figure 2.9 shows a block diagram representing Eq. (2.11) in the upper half, and a block diagram representing Eq. (2.12) in the lower half. This simple system can be analyzed using the linear analysis techniques that will be described in later chapters, or via Matlab as we saw in Example 2.1. It can also be numerically evaluated for an arbitrary input time history using Simulink, which is a sister software package to Matlab for interactive, nonlinear simulation and has a graphical user interface with drag and drop properties. Figure 2.10 shows a block diagram of the system as depicted by Simulink.

In many cases a system, such as the satellite shown in Fig. 2.7, has some flexibility in the structure. Depending on the nature of the flexibility, it can cause challenges in the design of a control system. Particular difficulty arises when there is flexibility between the sensor and actuator locations. Therefore, it is often important to include this flexibility in the model even when the system seems to be quite rigid.

EXAMPLE 2.4 *Flexibility: Flexible Satellite Attitude Control*

Figure 2.11(a) shows the situation where there is some flexibility between the satellite attitude sensor (θ_2) and the body of the satellite (θ_1) where the actuators are placed. Find the equations of motion and transfer function relating the motion of the instrument package to a control torque applied to the body of the satellite. For comparison, also determine the transfer function between the control torque to the attitude of the body of the satellite as if the sensors were located there. Retain the fexible model of the overall satellite for this second case, however.

Solution. The dynamic model for this situation is shown schematically in Fig. 2.11(b). This model is dynamically similar to the resonant system shown in Fig. 2.5, and results in equations of motion that are similar in form to Eqs. (2.8) and (2.9). The moments on each body are shown in the free-body diagrams in Fig. 2.12. The discussion of the moments on each body is essentially the same as the discussion for Example 2.2,

Figure 2.11
Model of the flexible satellite

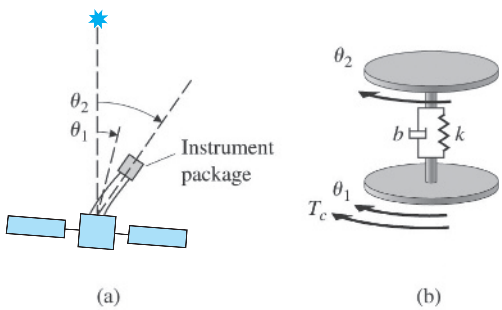
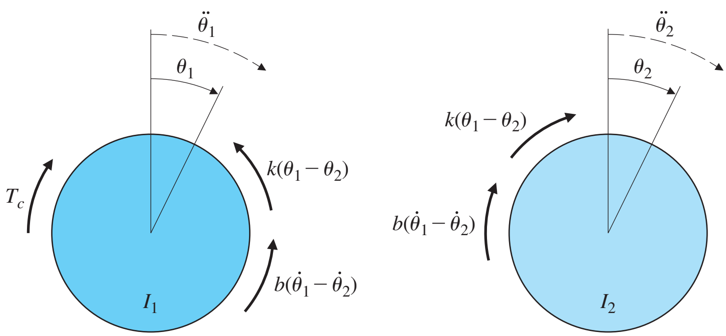


Figure 2.12
Free-body diagrams of the flexible satellite



except the springs and damper in that case produced forces, instead of moments that act on each inertia, as in this case. When the moments are summed, equated to the accelerations according to Eq. (2.10), and rearranged, the result is

$$I_1\ddot{\theta}_1 + b(\dot{\theta}_1 - \dot{\theta}_2) + k(\theta_1 - \theta_2) = T_c$$
$$I_2\ddot{\theta}_2 + b(\dot{\theta}_2 - \dot{\theta}_1) + k(\theta_2 - \theta_1) = 0.$$

Ignoring the damping b for simplicity, and substituting s for d/dt in the differential equations as we did for Example 2.2 yields

$$(I_1 s^2 + k)\Theta_1(s) - k\Theta_2(s) = T_c$$
$$-k\Theta_1(s) \;\; + (I_2 s^2 + k)\Theta_2(s) = 0.$$

Non-collocated sensor and actuator

Using Cramer's Rule as we did for Example 2.2, we find the transfer function between the control torque, T_c, and the sensor angle, θ_2, to be

$$\frac{\Theta_2(s)}{T_c(s)} = \frac{k}{I_1 I_2 s^2\left(s^2 + \frac{k}{I_1} + \frac{k}{I_2}\right)}. \tag{2.13}$$

For the second case, where we assume the attitude sensor is on the main body of the satellite, we want the transfer function between the control torque, T_c, and the satellite body angle, θ_1. Using Cramer's Rule again, we find that

$$\frac{\Theta_1(s)}{T_c(s)} = \frac{I_2 s^2 + k}{I_1 I_2 s^2\left(s^2 + \frac{k}{I_1} + \frac{k}{I_2}\right)}. \tag{2.14}$$

Collocated sensor and actuator

These two cases are typical of many situations in which the sensor and actuator may or may not be placed in the same location in a flexible body. We refer to the situation between sensor and actuator in Eq. (2.13) as the "noncollocated" case, whereas Eq. (2.14) describes the "collocated" case. You will see in Chapter 5 that it is far more difficult to control a system when there is flexibility between the sensor and actuator (noncollocated case) than when the sensor and actuator are rigidly attached to one another (the collocated case).

EXAMPLE 2.5 *Rotational Motion: Quadrotor Drone*

Figure 2.13 shows a small drone with four rotors. Find the equations of motion between an appropriate command to the individual motors and the three degrees of freedom; that is, pitch, roll, and yaw as defined by Fig. 2.14. The x and y axes are in the horizontal plane, while the z-axis is straight down. For this example, we only wish to describe the situation for very small motion about the initially level position of the coordinate system shown in Fig. 2.14. Note rotors 1 and 3 are rotating clockwise (CW) and rotors 2 and 4 are rotating counter clockwise (CCW); therefore, rotors 1 and 3 have an angular momentum in the

Figure 2.13
Quadcopter with a camera
Source: narongpon chaibot/Shutterstock

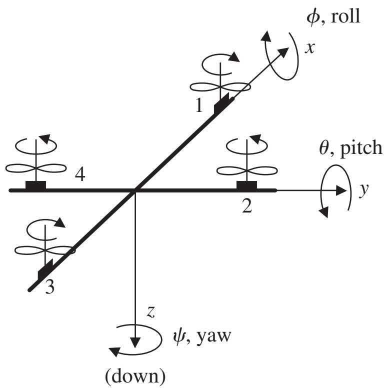


Figure 2.14
Orientation of the four rotors and definition of the attitude angles

$+z$ direction, while rotors 2 and 4 have an angular velocity in the $-z$ direction. Also, define what the "appropriate" commands would be in order to only produce the desired angular motion of pitch, roll, or yaw, without disturbing the other axes. The equations of motion for larger motions are complex and involve coupling between the axes as well as nonlinear terms due to angular motion, inertia asymmetry, and aerodynamics. These terms will be discussed in Chapter 10.

Solution. First, we need to establish what the commands should be to the motors attached to each of the four rotor blades in order to produce the desired motion without producing any undesired motion in another axis. Let's define the torque to each rotor as T_1, T_2, T_3, T_4. In steady hovering flight, there will be a torque applied to each rotor that maintains a steady rotor speed and thus a constant lift. The rotor speed stays constant because the torque from the motor just balances the aerodynamic drag on the rotor. If we were to add a perturbation that increased the torque magnitude applied to a rotor, the angular speed would increase until it reached a new equilibrium with the drag, and the rotor would produce an increased amount of lift. Likewise, for a negative perturbation in the torque magnitude on a rotor, the speed of the rotor and the lift would decrease. Note rotors 1 and 3 are rotating in a positive (CW) direction, hence there will be positive torques (T_1 and T_3) applied to those rotors, and negative torques (T_2 and T_4) applied to rotors 2 and 4 to maintain their negative (CCW) rotation. Another important aspect of this arrangement results from Newton's Third Law, that is: **For every action, there is an equal and opposite reaction.** This law tells us there are equal and opposite torques applied on the motors. Thus there are negative torques being applied to the 1 and 3 motors, while there are positive torques being applied to the 2 and 4 motors. In steady hovering flight, the torques being applied to the motors are all of equal magnitude and the two positive torques cancel out the two negative torques, hence the body of the quadrotor has no net torque applied about the z-axis and there is no yaw motion produced. (This is not the case for a single rotor helicopter where there is a large reaction torque applied to the engine, and that torque must be balanced by the tail rotor mounted perpendicular to the large lift rotor on top.)

To produce a control action to increase pitch, θ, without producing a torque about the other two axes, it makes sense to apply a small increase to the torque on rotor 1 with an equally small decrease to the torque on rotor 3. Thus, there is no net increase in the overall lift on the drone, and there is no change in the balance of the torques on the rotors nor their reaction torques on the drone itself. However, the positive change in lift from rotor 1 coupled with the negative change in lift from rotor 3 will produce a positive torque about the y-axis which will act to increase θ. Therefore, we produce the control torque for positive θ motion, T_θ, by setting $\delta T_1 = +T_\theta$ and $\delta T_3 = -T_\theta$. Following Example 2.3, the transfer function for pitch is

$$\frac{\Theta(s)}{T_\theta(s)} = \frac{1}{I_y}\frac{1}{s^2}. \tag{2.15}$$

Similarly, for roll control, we produce a positive roll torque, T_ϕ, by setting $\delta T_4 = -T_\phi$, thus increasing the negative rotation rate for rotor 4 and increasing it's resulting lift. Furthermore, we set $\delta T_2 = +T_\phi$, which reduces the lift from rotor 2, thus keeping the overall lift constant and contributing to the desired roll torque. The resulting transfer function for roll is

$$\frac{\Phi(s)}{T_\phi(s)} = \frac{1}{I_x}\frac{1}{s^2}. \tag{2.16}$$

Positive yaw control is accomplished by increasing the torque magntitude on rotors 2 and 4, while decreasing the torque magnitude on rotors 1 and 3 an equal amount. This will increase the lift from rotors 2 and 4 while decreasing the lift on rotors 1 and 3, thus producing no net change in the lift nor a torque that would influence θ or ϕ. But, the reaction torques will be in the positive direction for all four motors! This comes about because rotors 1 and 3 are rotating in a CW (positive direction) so a decrease in the torque applied to their rotors is a negative perturbation, thus resulting in positive reaction torques on the motors. Rotors 2 and 4 are rotating in a CCW (negative direction) so an increase in the torque magntitude applied to their rotors is also a negative perturbation, thus adding to the positive reaction torque applied to the motors. Therefore, the control torque for positive ψ motion, T_ψ, is produced by setting $\delta T_1 = \delta T_2 = \delta T_3 = \delta T_4 = -T_\psi$. The resulting transfer function is

$$\frac{\Psi(s)}{T_\psi(s)} = \frac{1}{I_z}\frac{1}{s^2}. \tag{2.17}$$

These three equations assume there is small motion from the horizontal orientation and thus any damping from aerodynamic forces are assumed negligible and the equations remain linear.

This example shows why quadrotors have become so popular for small drones; it is a well-balanced simple arrangement and does not require any complex mechanical arrangements to balance the torques. All the control can be accomplished by simply controlling the torque to the four rotors. Furthermore, with the definitions developed above for the motor commands and repeated here,

$$\text{For pitch;} \quad \delta T_1 = +T_\theta; \qquad \delta T_3 = -T_\theta, \tag{2.18}$$

$$\text{For roll;} \quad \delta T_2 = +T_\phi; \qquad \delta T_4 = -T_\phi, \tag{2.19}$$

$$\text{For yaw;} \quad \delta T_1 = \delta T_2 = \delta T_3 = \delta T_4 = -T_\psi, \tag{2.20}$$

the dynamics for each degree of attitude motion is uncoupled from the motion in the other axes.

In the special case in which a point in a rotating body is fixed with respect to an inertial reference frame, as is the case with a pendulum, Eq. (2.10) can be applied such that M is the sum of all moments about the *fixed* point, and I is the moment of inertia about the fixed point.

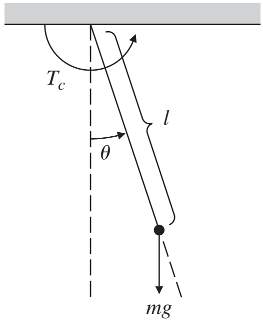
EXAMPLE 2.6 *Rotational Motion: Pendulum*

1. Write the equations of motion for the simple pendulum shown in Fig. 2.15, where all the mass is concentrated at the end point and there is a torque, T_c, applied at the pivot.
2. Use Matlab to determine the time history of θ to a step input in T_c of 1 N·m. Assume $l = 1$ m, $m = 1$ kg, and $g = 9.81$ m/sec^2.

Solution

1. **Equations of motion:** The moment of inertia about the pivot point is $I = ml^2$. The sum of moments about the pivot point contains a

Figure 2.15
Pendulum



term from gravity as well as the applied torque T_c. The equation of motion, obtained from Eq. (2.10), is

$$T_c - mgl\sin\theta = I\ddot{\theta}, \tag{2.21}$$

which is usually written in the form

$$\ddot{\theta} + \frac{g}{l}\sin\theta = \frac{T_c}{ml^2}. \tag{2.22}$$

This equation is nonlinear due to the $\sin\theta$ term. A general discussion of nonlinear equations will be contained in Chapter 9; however, we can proceed with a linearization of this case by assuming the motion is small enough that $\sin\theta \cong \theta$. Then, Eq. (2.22) becomes the linear equation

$$\ddot{\theta} + \frac{g}{l}\theta = \frac{T_c}{ml^2}. \tag{2.23}$$

With no applied torque, the natural motion is that of a harmonic oscillator with a natural frequency of[4]

$$\omega_n = \sqrt{\frac{g}{l}}. \tag{2.24}$$

The transfer function can be obtained as described for Eq. (2.7), yielding

$$\frac{\Theta(s)}{T_c(s)} = \frac{\frac{1}{ml^2}}{s^2 + \frac{g}{l}}. \tag{2.25}$$

2. **Time history:** The dynamics of a system can be prescribed to Matlab in terms of its transfer function and the step response via the step function. The Matlab statements

[4] In a grandfather clock, it is desired to have a pendulum period of exactly 2 sec. Show that the pendulum should be approximately 1 m in length.

```
t = 0:0.02:10;            % vector of times for output, 0 to 10 at 0.02
                            increments
m = 1;                    % value of mass (Kg)
L = 1;                    % value of length (m)
g = 9.81;                 % value of gravity, g (m/sec^2)
s = tf('s');              % sets up transfer function input mode
sys = (1/(m*L^2))/
      (s^2 + g/L)
y = step(sys,t);          % computes step responses at times given
                            by t for step at t = 0
Rad2Deg = 57.3;           % converts radians to degrees
plot(t, Rad2Deg*y)        % converts output from radians to degrees
                            and plots step response
```

will produce the desired time history shown in Fig. 2.16.

As we saw in this example, the resulting equations of motion are often nonlinear. Such equations are much more difficult to solve than linear ones, and the kinds of possible motions resulting from a nonlinear model are much more difficult to categorize than those resulting from a linear model. It is therefore useful to linearize models in order to gain access to linear analysis methods. It may be that the linear models and linear analysis are used only for the design of the control system (whose function may be to maintain the system in the linear region). Once a control system is synthesized and shown to have desirable performance based on linear analysis, it is then prudent to carry out further analysis or an accurate numerical simulation of the system with the significant nonlinearities in order to validate that performance. **Simulink** is an expedient way to carry out these simulations and can handle most

Simulink

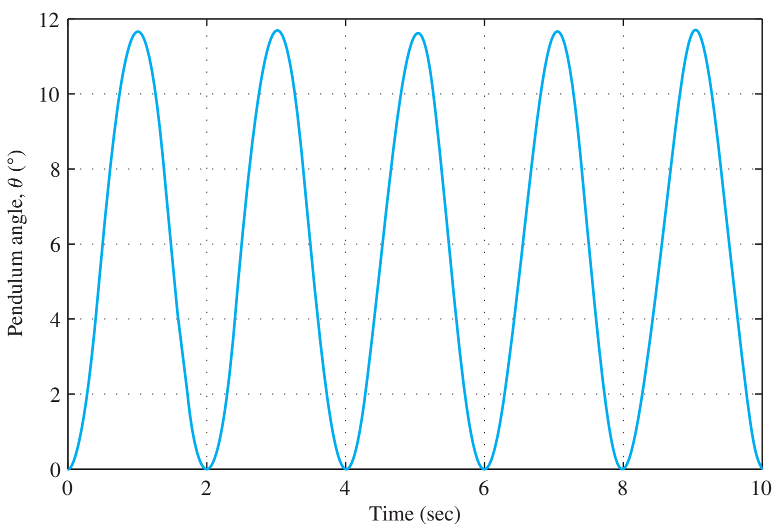


Figure 2.16
Response of the pendulum to a step input of 1 N·m in the applied torque

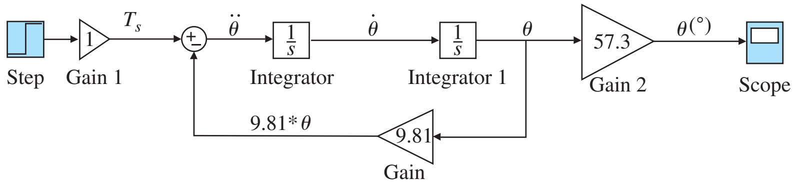


Figure 2.17
The Simulink block diagram representing the linear equation (2.26)

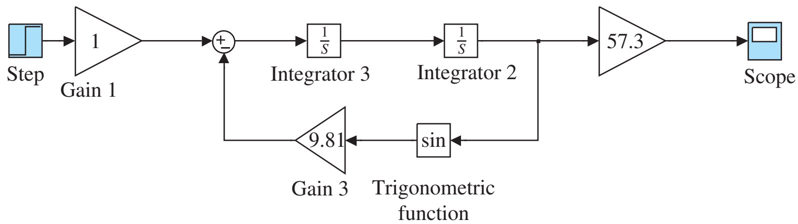


Figure 2.18
The Simulink block diagram representing the nonlinear equation (2.27)

nonlinearities. Use of this simulation tool is carried out by constructing a block diagram[5] that represents the equations of motion. The linear equation of motion for the pendulum with the parameters as specified in Example 2.6 can be seen from Eq. (2.23) to be

$$\ddot{\theta} = -9.81 * \theta + 1, \tag{2.26}$$

and this is represented in Simulink by the block diagram in Fig. 2.17. Note the circle on the left side of the figure with the + and − signs indicating addition and subtraction, implements Eq. (2.26).

The result of running this numerical simulation will be essentially identical to the linear solution shown in Fig. 2.16 because the solution is for relatively small angles where $\sin\theta \cong \theta$. However, using Simulink to solve for the response enables us to simulate the nonlinear equation so we could analyze the system for larger motions. In this case, Eq. (2.26) becomes

$$\ddot{\theta} = -9.81 * \sin\theta + 1, \tag{2.27}$$

and the Simulink block diagram shown in Fig. 2.18 implements this nonlinear equation.

Simulink is capable of simulating all commonly encountered nonlinearities, including deadzones, on–off functions, stiction, hysteresis, aerodynamic drag (a function of v^2), and trigonometric functions. All real systems have one or more of these characteristics in varying degrees. These nonlinearities will be expanded upon in detail in Chapter 9.

EXAMPLE 2.7 *Use of Simulink for Nonlinear Motion: Pendulum*

Use Simulink to determine the time history of θ for the pendulum in Example 2.6. Compare it against the linear solution for T_c values of 1 N·m and 4 N·m.

[5] A more extensive discussion of block diagrams is contained in Section 3.2.1 of Chapter 3

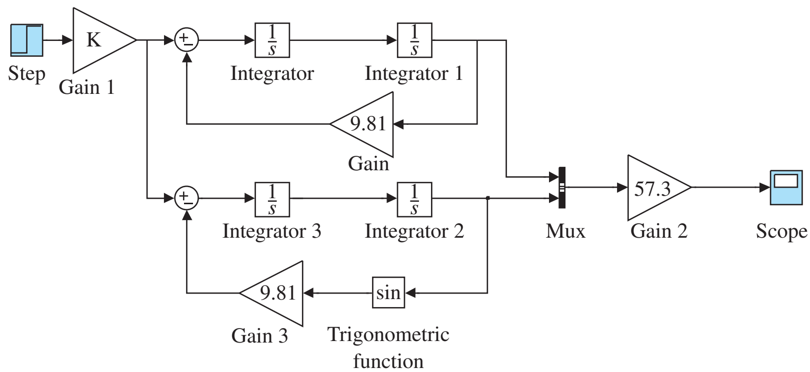


Figure 2.19
Block diagram of the pendulum for both the linear and nonlinear models

Solution. Time history: The Simulink block diagrams for the two cases discussed above are combined and both outputs in Figs. 2.17 and 2.18 are sent via a "multiplexer block (Mux)" to the "scope" so they can be plotted on the same graph. Figure 2.19 shows the combined block diagram where the gain, K, represents the values of T_c. The outputs of this system for T_c values of 1 N · m and 4 N· m are shown in Fig. 2.20. Note for $T_c = 1$ N·m, the outputs at the top of the figure remain at 12° or less, and the linear approximation is extremely close to the nonlinear output. For $T_c = 4$ N·m, the output angle grows near to 50° and a substantial difference in the response magnitude and frequency is apparent due to θ being a poor approximation to $\sin\theta$ at these magnitudes. In fact, since $\sin\theta$ compared to θ signifies a reduced gravitational restoring force at the higher angles, we see an increased amplitude and slower frequency.

Chapter 9 will be devoted to the analysis of nonlinear systems and greatly expands on these ideas.

2.1.3 Combined Rotation and Translation

In some cases, mechanical systems contain both translational and rotational portions. The procedure is the same as that described in Sections 2.1.1 and 2.1.2: sketch the free-body diagrams, define coordinates and positive directions, determine all forces and moments acting, and apply Eqs. (2.1) and/or (2.10). An exact derivation of the equations for these systems can become quite involved; therefore, the complete analysis for the following example is contained in Appendix W2.1.4 located at www.pearsonglobaleditions.com, and only the linearized equations of motion and their transfer functions are given here.

EXAMPLE 2.8 *Rotational and Translational Motion: Hanging Crane*

Write the equations of motion for the hanging crane shown schematically in Fig. 2.21. Linearize the equations about $\theta = 0$, which would typically be valid for the hanging crane. Also, linearize the equations for

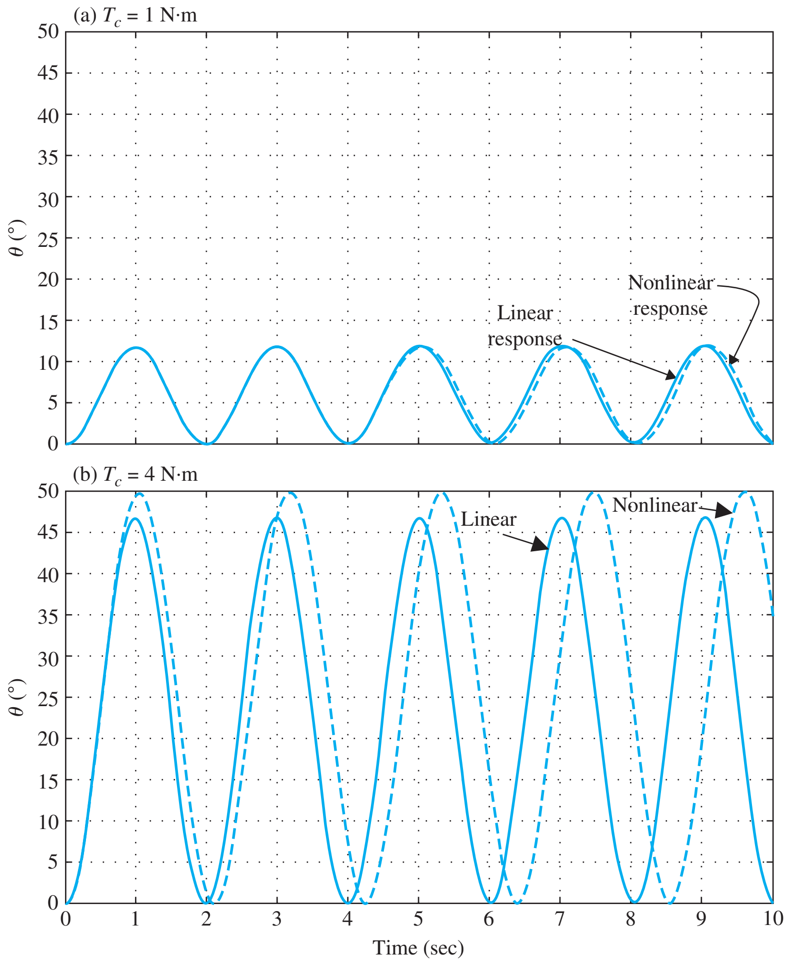


Figure 2.20
Response of the pendulum Simulink numerical simulation for the linear and nonlinear models: (a) for $T_c = 1$ N·m; (b) $T_c = 4$ N·m

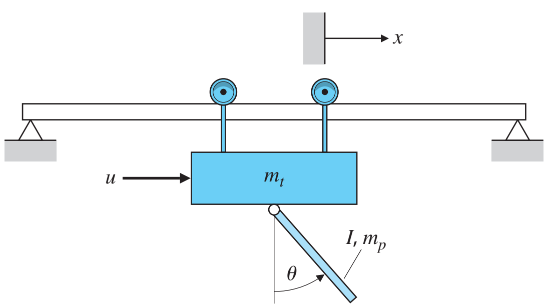


Figure 2.21
Schematic of the crane with hanging load

$\theta = \pi$, which represents the situation for the inverted pendulum shown in Fig. 2.22. The trolley has mass m_t and the hanging crane (or pendulum) has mass m_p and inertia about its mass center of I. The distance from the pivot to the mass center of the pendulum is l; therefore, the moment of inertia of the pendulum about the pivot point is $(I + m_p l^2)$.

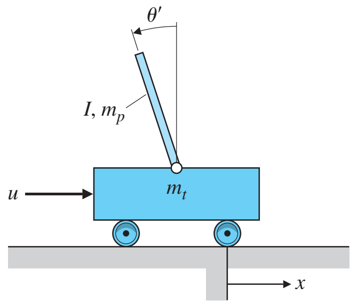


Figure 2.22
Inverted pendulum

Solution. Free-body diagrams need to be drawn for the trolley and the pendulum and the reaction forces considered where the two attach to one another. We carry out this process in Appendix W2.1.3. After Newton's laws are applied for the translational motion of the trolley and the rotational motion of the pendulum, it will be found that the reaction forces between the two bodies can be eliminated, and the only unknowns will be θ and x. The results are two coupled second-order nonlinear differential equations in θ and x with the input being the force applied to the trolley, u. They can be linearized in a manner similar to that done for the simple pendulum by assuming small angles. For small motions about $\theta = 0$, we let $\cos\theta \cong 1$, $\sin\theta \cong \theta$, and $\dot{\theta}^2 \cong 0$; thus the equations are approximated by

$$(I + m_p l^2)\ddot{\theta} + m_p g l\theta = -m_p l\ddot{x},$$

$$(m_t + m_p)\ddot{x} + b\dot{x} + m_p l\ddot{\theta} = u. \tag{2.28}$$

Note the first equation is very similar to the simple pendulum, Eq. (2.21), where the applied torque arises from the trolley accelerations. Likewise, the second equation representing the trolley motion, x, is very similar to the car translation in Eq. (2.3), where the forcing term arises from the angular acceleration of the pendulum. Eliminating x in these two coupled equations leads to the desired transfer function. Neglecting the friction term, b, simplifies the algebra and leads to an approximate transfer function from the control input u to hanging crane angle θ:

$$\frac{\Theta(s)}{U(s)} = \frac{-m_p l}{((I + m_p l^2)(m_t + m_p) - m_p^2 l^2)s^2 + m_p g l(m_t + m_p)}. \tag{2.29}$$

Inverted pendulum equations

For the inverted pendulum in Fig. 2.22, where $\theta \cong \pi$, assume $\theta = \pi + \theta'$, where θ' represents motion from the vertical *upward* direction. In this case, $\sin\theta \cong -\theta'$, $\cos\theta \cong -1$, and the nonlinear equations become[6]

[6]The inverted pendulum is often described with the angle of the pendulum being positive for *clockwise* motion. If defined that way, then the sign reverses on all terms in Eqs. (2.30) in θ' or $\ddot{\theta}'$.

$$(I + m_p l^2)\ddot{\theta}' - m_p g l \theta' = m_p l \ddot{x},$$
$$(m_t + m_p)\ddot{x} + b\dot{x} - m_p l \ddot{\theta}' = u. \tag{2.30}$$

As noted in Example 2.2, a stable system will always have the same signs on each variable, which is the case for the stable hanging crane modeled by Eqs. (2.28). However, the signs on θ and $\ddot{\theta}$ in the first equation in Eq. (2.30) are opposite, thus indicating instability, which is the characteristic of the inverted pendulum.

The transfer function, again without friction, is

$$\frac{\Theta'(s)}{U(s)} = \frac{m_p l}{((I + m_p l^2)(m_t + m_p) - m_p^2 l^2)s^2 - m_p g l(m_t + m_p)}. \tag{2.31}$$

Evaluation of this transfer function for an infinitessimal step in u will result in a diverging value of θ' thus requiring feedback to remain upright, a subject for Chapter 5.

In Chapter 5, you will learn how to stabilize systems using feedback and will see that even unstable systems like an inverted pendulum can be stabilized provided there is a sensor that measures the output quantity and a control input. For the case of the inverted pendulum perched on a trolley, it would be required to measure the pendulum angle, θ', and provide a control input, u, that accelerated the trolley in such a way that the pendulum remained pointing straight up. In years past, this system existed primarily in university control system laboratories as an educational tool. However, more recently, there is a practical device in production and being sold that employs essentially this same dynamic system: the Segway. It uses a gyroscope so the angle of the device is known with respect to vertical, and electric motors provide a torque on the wheels so it balances the device and provides the desired forward or backward motion. It is shown in Fig. 2.23.

2.1.4 Complex Mechanical Systems

This section contains the derivation of the equations of motion for mechanical systems. In particular, it contains the full derivation of the equations of motion for the hanging crane in Example 2.8 and the inverted pendulum on a cart. See Appendix W2.1.4 at www.pearsonglobaleditions.com.

2.1.5 Distributed Parameter Systems

All the preceding examples contained one or more rigid bodies, although some were connected to others by springs. Actual structures—for example, satellite solar panels, airplane wings, or robot arms—usually bend, as shown by the flexible beam in Fig. 2.24(a). The equation describing its motion is a fourth-order *partial* differential equation that arises because the mass elements are continuously distributed along the beam with a small amount of flexibility between

Figure 2.23
The Segway, which is similar to the inverted pendulum and is kept upright by a feedback control system
Source: Photo courtesy of David Powell

elements. This type of system is called a **distributed parameter system**. The dynamic analysis methods presented in this section are not sufficient to analyze this case; however, more advanced texts (Thomson and Dahleh, 1998) show the result is

$$EI\frac{\partial^4 w}{\partial x^4} + \rho\frac{\partial^2 w}{\partial t^2} = 0, \tag{2.32}$$

where

E = Young's modulus,

I = beam area moment of inertia,

ρ = beam density,

w = beam deflection at length x along the beam.

The exact solution to Eq. (2.32) is too cumbersome to use in designing control systems, but it is often important to account for the gross effects of bending in control systems design.

The continuous beam in Fig. 2.24(b) has an infinite number of vibration-mode shapes, all with different frequencies. Typically, the lowest-frequency modes have the largest amplitude and are the most

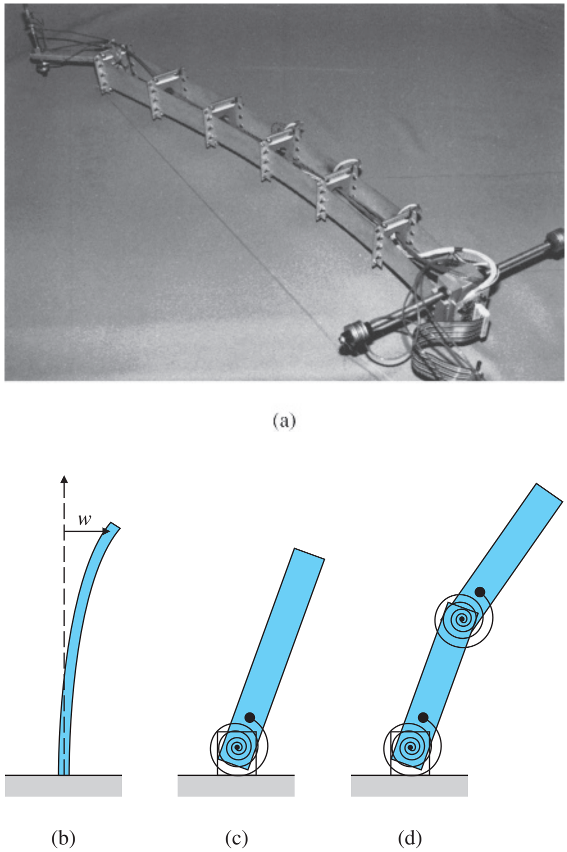


Figure 2.24
(a) Flexible robot arm used for research at Stanford University; (b) model for continuous flexible beam; (c) simplified model for the first bending mode; (d) model for the first and second bending modes

Source: Photo courtesy of E. Schmitz

important to approximate well. The simplified model in Fig. 2.24(c) can be made to duplicate the essential behavior of the first bending mode shape and frequency, and would usually be adequate for controller design. If frequencies higher than the first bending mode are anticipated in the control system operation, it may be necessary to model the beam as shown in Fig. 2.24(d), which can be made to approximate the first two bending modes and frequencies. Likewise, higher-order models can be used if such accuracy and complexity are deemed necessary (Schmitz, 1985; Thomson and Dahleh, 1998). When a continuously bending object is approximated as two or more rigid bodies connected by springs, the resulting model is sometimes referred to as a **lumped parameter model**.

A flexible structure can be approximated by a lumped parameter model

2.1.6 Summary: Developing Equations of Motion for Rigid Bodies

The physics necessary to write the equations of motion of a rigid body is entirely given by Newton's laws of motion. The method is as follows:

1. Assign variables such as x and θ that are both necessary and sufficient to describe an *arbitrary* position of the object.
2. Draw a free-body diagram of each component. Indicate *all* forces acting on each body and their reference directions. Also indicate the accelerations of the center of mass with respect to an inertial reference for each body.
3. Apply Newton's law in translation [Eq. (2.1)] and/or rotation [Eq. (2.10)] form.
4. Combine the equations to eliminate internal forces.
5. The number of independent equations should equal the number of unknowns.

2.2 Models of Electric Circuits

Electric circuits are frequently used in control systems largely because of the ease of manipulation and processing of electric signals. Although controllers are increasingly implemented with digital logic, many functions are still performed with analog circuits. Analog circuits are faster than digital and, for very simple controllers, an analog circuit would be less expensive than a digital implementation. Furthermore, the power amplifier for electromechanical control and the anti-alias prefilters for digital control must be analog circuits.

Electric circuits consist of interconnections of sources of electric voltage and current, and other electronic elements such as resistors, capacitors, and transistors. An important building block for circuits is an operational amplifier (or op-amp),[7] which is also an example of a complex feedback system. Some of the most important methods of feedback system design were developed by the designers of high-gain, wide-bandwidth feedback amplifiers, mainly at the Bell Telephone Laboratories between 1925 and 1940. Electric and electronic components also play a central role in electromechanical energy conversion devices such as electric motors, generators, and electrical sensors. In this brief survey, we cannot derive the physics of electricity or give a comprehensive review of all the important analysis techniques. We will define the variables, describe the relations imposed on them by typical elements and circuits, and describe a few of the most effective methods available for solving the resulting equations.

Symbols for some linear circuit elements and their current–voltage relations are given in Fig. 2.25. Passive circuits consist of interconnections of resistors, capacitors, and inductors. With electronics, we increase the set of electrical elements by adding active devices, including diodes, transistors, and amplifiers.

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[7] Oliver Heaviside introduced the mathematical operation p to signify differentiation so that $pv = dv/dt$. The Laplace transform incorporates this idea, using the complex variable s. Ragazzini et al. (1947) demonstrated that an ideal, high-gain electronic amplifier permitted one to realize arbitrary "operations" in the Laplace transform variable s, so they named it the operational amplifier, commonly abbreviated to op-amp.

Kirchhoff's laws

The basic equations of electric circuits, called Kirchhoff's laws, are as follows:

1. **Kirchhoff's current law (KCL).** The algebraic sum of currents leaving a junction or node equals the algebraic sum of currents entering that node.
2. **Kirchhoff's voltage law (KVL).** The algebraic sum of all voltages taken around a closed path in a circuit is zero.

With complex circuits of many elements, it is essential to write the equations in a careful, well-organized way. Of the numerous methods for doing this, we choose for description and illustration the popular and powerful scheme known as **node analysis**. One node is selected as a reference and we assume the voltages of all other nodes to be unknowns. The choice of reference is arbitrary in theory, but in actual electronic circuits the common, or ground, terminal is the obvious and standard choice. Next, we write equations for the selected unknowns using the current law (KCL) at each node. We express these currents in terms of the selected unknowns by using the element equations in Fig. 2.25. If the circuit contains voltage sources, we must substitute a voltage law (KVL) for such sources. Example 2.9 illustrates how node analysis works.

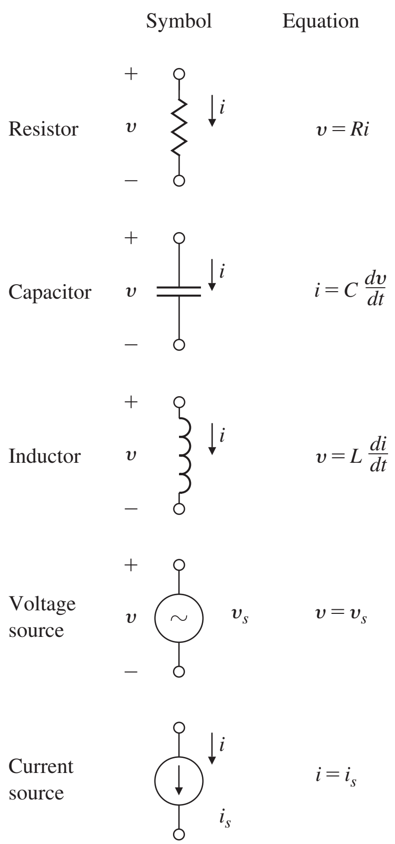


Figure 2.25
Elements of electric circuits

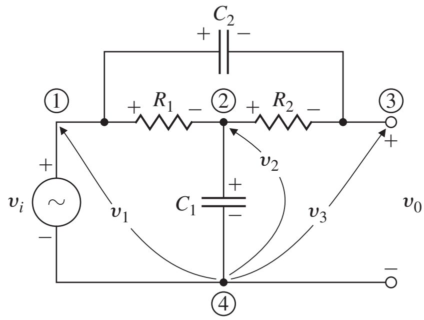


Figure 2.26
Bridged tee circuit

EXAMPLE 2.9 *Equations for the Bridged Tee Circuit*

Determine the differential equations for the circuit shown in Fig. 2.26.

Solution. We select node 4 as the reference and the voltages v_1, v_2, and v_3 at nodes 1, 2, and 3 as the unknowns. We start with the degenerate KVL relationship

$$v_1 = v_i. \tag{2.33}$$

At node 2, the KCL is

$$-\frac{v_1 - v_2}{R_1} + \frac{v_2 - v_3}{R_2} + C_1\frac{dv_2}{dt} = 0, \tag{2.34}$$

and at node 3, the KCL is

$$\frac{v_3 - v_2}{R_2} + C_2\frac{d(v_3 - v_1)}{dt} = 0. \tag{2.35}$$

These three equations describe the circuit. If desired, one could eliminate v_2 from the above equations, thus obtaining a second-order differential equation that describes the dynamic relationship between the input, $v_i(= v_1)$, and output, $v_3(= v_o)$.

EXAMPLE 2.10 *Equations for a Circuit with a Current Source*

Determine the differential equations for the circuit shown in Fig. 2.27. Choose the capacitor voltages and the inductor current as the unknowns.

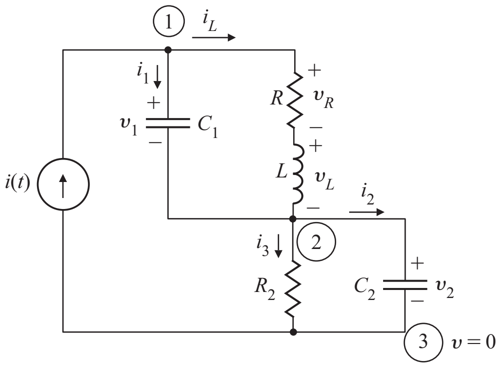
Solution. We select node 3 as the reference and the voltages v_1 and v_2, and the current through the inductor, i_L, as unknowns. We start the KCL relationships: relationships:
At node 1, the KCL is

$$i(t) = i_L + i_1, \tag{2.36}$$

and at node 2, the KCL is

$$i_L + i_1 = i_2 + i_3. \tag{2.37}$$

Figure 2.27
Circuit for Example 2.10



Furthermore, from Fig. 2.27, we see that

$$i_3 = \frac{v_2}{R_2}, \tag{2.38}$$

$$i_1 = C_1 \frac{dv_1}{dt}, \tag{2.39}$$

$$i_2 = C_2 \frac{dv_2}{dt}, \tag{2.40}$$

$$v_R = i_L R, \tag{2.41}$$

$$L\frac{di_L}{dt} = v_1 - v_R. \tag{2.42}$$

These reduce to three differential equations in the three unknowns,

$$L\frac{di_L}{dt} = v_1 - i_L R, \tag{2.43}$$

$$C_1\frac{dv_1}{dt} = i(t) - i_L, \tag{2.44}$$

$$C_2\frac{dv_2}{dt} = i(t) - \frac{v_2}{R_2}. \tag{2.45}$$

Operational amplifier

Kirchhoff's laws can also be applied to circuits that contain an **operational amplifier**. The simplified circuit of the op-amp is shown in Fig. 2.28(a) and the schematic symbol is drawn in Fig. 2.28(b). If the positive terminal is not shown, it is assumed to be connected to ground, $v_+ = 0$, and the reduced symbol of Fig. 2.28(c) is used. For use in control circuits, it is usually assumed that the op-amp is *ideal* with the values $R_1 = \infty$, $R_0 = 0$, and $A = \infty$. The equations of the ideal op-amp are extremely simple, being

$$i_+ = i_- = 0, \tag{2.46}$$

$$v_+ - v_- = 0. \tag{2.47}$$

The gain of the amplifier is assumed to be so high that the output voltage becomes v_{out} = *whatever it takes* to satisfy these equations. Of

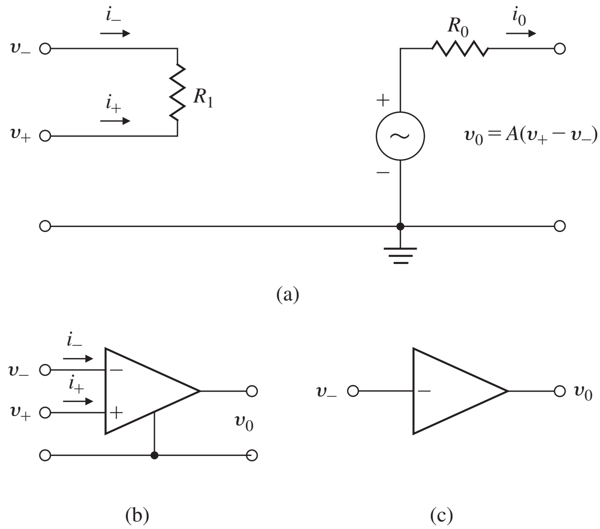


Figure 2.28
(a) Op-amp simplified circuit; (b) op-amp schematic symbol; (c) reduced symbol for $v_+ = 0$

course, a real amplifier only approximates these equations, but unless they are specifically described, we will assume all op-amps are ideal. More realistic models are the subject of several problems given at the end of the chapter.

EXAMPLE 2.11 *Op-Amp Summer*

Find the equations and transfer functions of the circuit shown in Fig. 2.29.

Solution. Equation (2.47) requires that $v_- = 0$, and thus the currents are $i_1 = v_1/R_1$, $i_2 = v_2/R_2$, and $i_{out} = v_{out}/R_f$. To satisfy Eq. (2.46), $i_1 + i_2 + i_{out} = 0$, from which it follows that $v_1/R_1 + v_2/R_2 + v_{out}/R_f = 0$, and we have

$$v_{out} = -\left[\frac{R_f}{R_1}v_1 + \frac{R_f}{R_2}v_2\right]. \tag{2.48}$$

The op-amp summer

From this equation, we see the circuit output is a weighted sum of the input voltages with a sign change. The circuit is called a **summer**.

A second important example for control is given by the op-amp integrator.

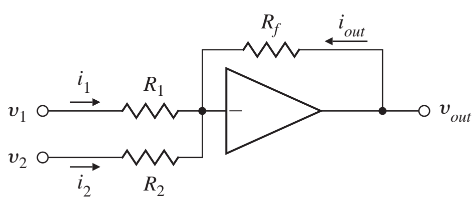


Figure 2.29
The op-amp summer

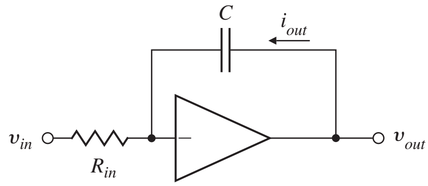


Figure 2.30
The op-amp integrator

EXAMPLE 2.12 *Integrator*

Op-amp as integrator

Find the transfer function for the circuit shown in Fig. 2.30.

Solution. In this case, the equations are differential and Eqs. (2.46) and (2.47) require

$$i_{in} + i_{out} = 0, \tag{2.49}$$

so

$$\frac{v_{in}}{R_{in}} + C\frac{dv_{out}}{dt} = 0. \tag{2.50}$$

Equation (2.50) can be written in integral form as

$$v_{out} = -\frac{1}{R_{in}C}\int_0^t v_{in}(\tau)\,d\tau + v_{out}(0). \tag{2.51}$$

Using the operational notation that $d/dt = s$ in Eq. (2.50), the transfer function (which assumes zero initial conditions) can be written as

$$V_{out}(s) = -\frac{1}{s}\frac{V_{in}(s)}{R_{in}C}. \tag{2.52}$$

Thus the ideal op-amp in this circuit performs the operation of integration and the circuit is simply referred to as an **integrator**.

2.3 Models of Electromechanical Systems

Electric current and magnetic fields interact in two ways that are particularly important to an understanding of the operation of most electromechanical actuators and sensors. If a current of i amp in a conductor of length l m is arranged at right angles in a magnetic field of B teslas, then there is a force on the conductor at right angles to the plane of i and B, with magnitude

$$F = Bli \text{ N}. \tag{2.53}$$

Law of motors

This equation is the basis of conversion of electric energy to mechanical work and is called the **law of motors**.

2.3.1 Loudspeakers

EXAMPLE 2.13 *Modeling a Loudspeaker*

A typical geometry for a loudspeaker for producing sound is sketched in Fig. 2.31. The permanent magnet establishes a radial field in the cylindrical gap between the poles of the magnet. The force on the conductor

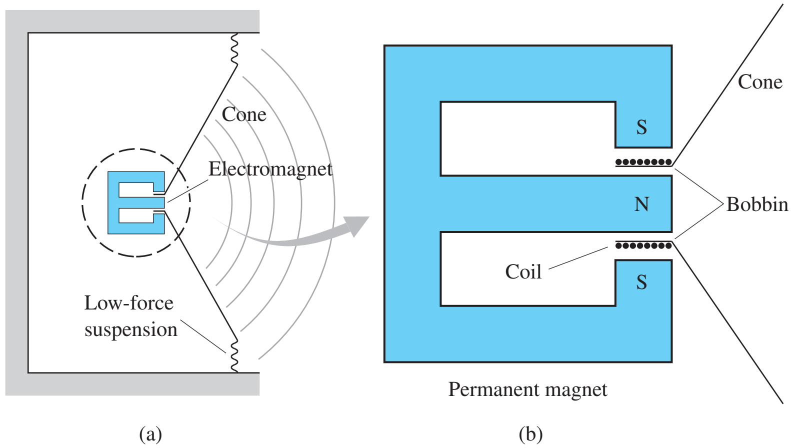


Figure 2.31
Geometry of a loudspeaker: (a) overall configuration; (b) the electromagnet and voice coil

wound on the bobbin causes the voice coil to move, producing sound. The effects of the air can be modeled as if the cone had equivalent mass M and viscous friction coefficient b. Assume the magnet establishes a uniform field B of 0.4 tesla and the bobbin has 18 turns at a 1.9-cm diameter. Write the equations of motion of the device.

Solution. The current is at right angles to the field, and the force of interest is at right angles to the plane of i and B, so Eq. (2.53) applies. In this case the field strength is $B = 0.4$ tesla and the conductor length is

$$l = 18 \times 2\pi \frac{0.95}{100} = 1.074 \text{ m}.$$

Thus, the force is

$$F = 0.4 \times 1.074 \times i = 0.43i \text{ N}.$$

The mechanical equation follows from Newton's laws, and for a mass M and friction coefficient b, the equation is

$$M\ddot{x} + b\dot{x} = 0.43i. \tag{2.54}$$

This second-order differential equation describes the motion of the loudspeaker cone as a function of the input current i driving the system. Substituting s for d/dt in Eq. (2.54) as before, the transfer function is easily found to be

$$\frac{X(s)}{I(s)} = \frac{\frac{0.43}{M}}{s\left(s + \frac{b}{M}\right)}. \tag{2.55}$$

The second important electromechanical relationship is the effect of mechanical motion on electric voltage. If a conductor of length l m is moving in a magnetic field of B teslas at a velocity of v m/sec at mutually right angles, an electric voltage is established across the conductor with magnitude

$$e = Blv \text{ V}. \tag{2.56}$$

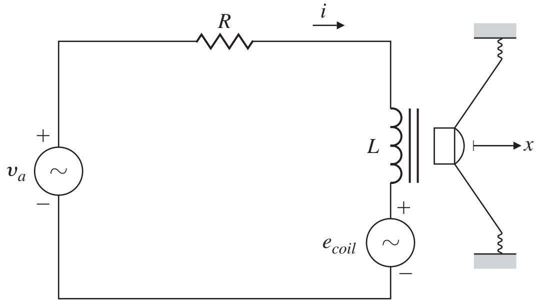


Figure 2.32
A loudspeaker showing the electric circuit

Law of generators

This expression is called the **law of generators.**

EXAMPLE 2.14 *Loudspeaker with Circuit*

For the loudspeaker in Fig. 2.31 and the circuit driving it in Fig. 2.32, find the differential equations relating the input voltage v_a to the output cone displacement x. Assume the effective circuit resistance is R and the inductance is L.

Solution. The loudspeaker motion satisfies Eq. (2.54), and the motion results in a voltage across the coil as given by Eq. (2.56), with the velocity $\dot{x}$. The resulting voltage is

$$e_{coil} = Bl\dot{x} = 0.43\dot{x}. \tag{2.57}$$

This induced voltage effect needs to be added to the analysis of the circuit. The equation of motion for the electric circuit is

$$L\frac{di}{dt} + Ri = v_a - 0.43\dot{x}. \tag{2.58}$$

These two coupled equations, (2.54) and (2.58), constitute the dynamic model for the loudspeaker.

Again, substituting s for d/dt in these equations and replacing all the parameters with the given numerical values, the transfer function between the applied voltage and the loudspeaker displacement is found to be

$$\frac{X(s)}{V_a(s)} = \frac{0.43}{s\left[(Ms+b)(Ls+R)+(0.43)^2\right]}. \tag{2.59}$$

2.3.2 Motors

DC motor actuators

A common actuator based on the laws of motors and generators and used in control systems is the direct current (DC) motor to provide rotarymotion. A sketch of the basic components of a DC motor is given in Fig. 2.33. In addition to housing and bearings, the nonturning part (stator) has magnets, which establish a field across the rotor. The magnets may be electromagnets or, for small motors, permanent magnets. The brushes contact the rotating commutator, which causes the current

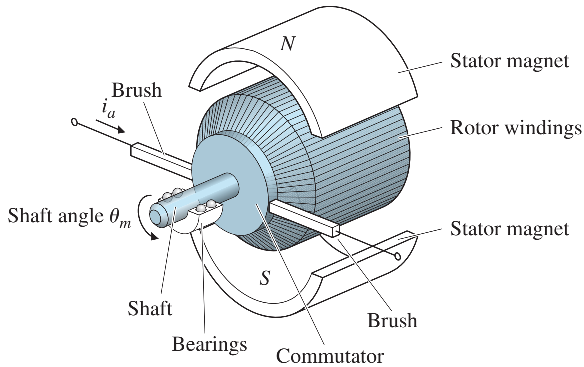


Figure 2.33
Sketch of a DC motor

always to be in the proper conductor windings so as to produce maximum torque. If the direction of the current is reversed, the direction of the torque is reversed.

Back emf

The motor equations give the torque T on the rotor in terms of the armature current i_a and express the back emf voltage in terms of the shaft's rotational velocity $\dot{\theta}_m$.[8]
Thus,

$$T = K_t\, i_a, \tag{2.60}$$

$$e = K_e\, \dot{\theta}_m. \tag{2.61}$$

In consistent units, the torque constant K_t equals the electric constant K_e, but in some cases, the torque constant will be given in other units, such as ounce-inches per ampere, and the electric constant may be expressed in units of volts per 1000 rpm. In such cases, the engineer must make the necessary translations to be certain the equations are correct.

EXAMPLE 2.15 *Modeling a DC Motor*

Find the equations for a DC motor with the equivalent electric circuit shown in Fig. 2.34(a). Assume the rotor has inertia J_m and viscous friction coefficient b.

Solution. The free-body diagram for the rotor, shown in Fig. 2.34(b), defines the positive direction and shows the two applied torques, T and $b\dot{\theta}_m$. Application of Newton's laws yields

$$J_m\ddot{\theta}_m + b\dot{\theta}_m = K_t i_a. \tag{2.62}$$

Analysis of the electric circuit, including the back emf voltage, shows the electrical equation to be

$$L_a\frac{di_a}{dt} + R_a i_a = v_a - K_e\dot{\theta}_m. \tag{2.63}$$

[8] Because the generated electromotive force (emf) works against the applied armature voltage, it is called the **back emf**.

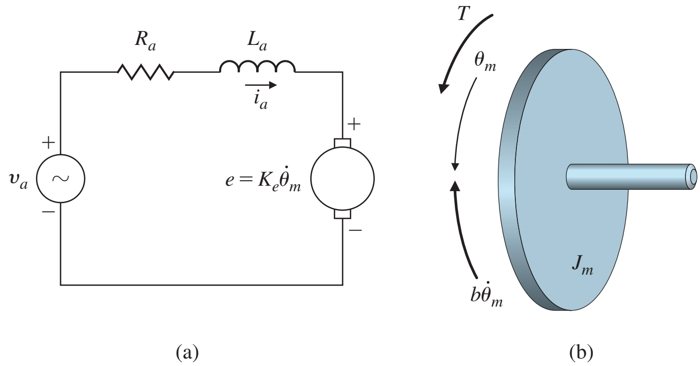


Figure 2.34
DC motor: (a) electric circuit of the armature; (b) free-body diagram of the rotor

With s substituted for d/dt in Eqs. (2.62) and (2.63), the transfer function for the motor is readily found to be

$$\frac{\Theta_m(s)}{V_a(s)} = \frac{K_t}{s[(J_m s + b)(L_a s + R_a) + K_t K_e]}. \tag{2.64}$$

In many cases the relative effect of the inductance is negligible compared with the mechanical motion and can be neglected in Eq. (2.63). If so, we can combine Eqs. (2.62) and (2.63) into one equation to get

$$J_m\ddot{\theta}_m + \left(b + \frac{K_t K_e}{R_a}\right)\dot{\theta}_m = \frac{K_t}{R_a}v_a. \tag{2.65}$$

From Eq. (2.65) it is clear that in this case the effect of the back emf is indistinguishable from the friction, and the transfer function is

$$\frac{\Theta_m(s)}{V_a(s)} = \frac{\frac{K_t}{R_a}}{J_m s^2 + \left(b + \frac{K_t K_e}{R_a}\right)s} \tag{2.66}$$

$$= \frac{K}{s(\tau s + 1)}, \tag{2.67}$$

where

$$K = \frac{K_t}{bR_a + K_t K_e}, \tag{2.68}$$

$$\tau = \frac{R_a J_m}{bR_a + K_t K_e}. \tag{2.69}$$

In many cases, a transfer function between the motor input and the output speed ($\omega = \dot{\theta}_m$) is required. In such cases, the transfer function would be

$$\frac{\Omega(s)}{V_a(s)} = s\frac{\Theta_m(s)}{V_a(s)} = \frac{K}{\tau s + 1}. \tag{2.70}$$

AC motor actuators

Another device used for electromechanical energy conversion is the alternating current (AC) induction motor invented by N. Tesla. Elementary analysis of the AC motor is more complex than that of the DC motor. A typical experimental set of curves of torque versus speed for fixed frequency and varying amplitude of applied (sinusoidal) voltage is given in Fig. 2.35. Although the data in the figure are for a constant engine speed, they can be used to extract the motor constants that will provide a dynamic model for the motor. For analysis of a control problem involving an AC motor such as that described by Fig. 2.35, we make a linear approximation to the curves for speed near zero and at a midrange voltage to obtain the expression

$$T = K_1 v_a - K_2 \dot{\theta}_m. \tag{2.71}$$

The constant K_1 represents the ratio of a change in torque to a change in voltage at zero speed, and is proportional to the distance between the curves at zero speed. The constant K_2 represents the ratio of a change in torque to a change in speed at zero speed and a midrange voltage; therefore, it is the slope of a curve at zero speed as shown by the line at V_2. For the electrical portion, values for the armature resistance R_a and inductance L_a are also determined by experiment. Once we have values for K_1, K_2, R_a, and L_a, the analysis proceeds as the analysis in Example 2.15 for the DC motor. For the case in which the inductor can be neglected, we can substitute K_1 and K_2 into Eq. (2.65) in place of K_t/R_a and $K_t K_e/R_a$, respectively.

In addition to the DC and AC motors mentioned here, control systems use brushless DC motors (Reliance Motion Control Corp., 1980) and stepping motors (Kuo, 1980). Models for these machines, developed in the works just cited, do not differ in principle from the motors considered in this section. In general, the analysis, supported by experiment, develops the torque as a function of voltage and speed similar to the AC motor torque–speed curves given in Fig. 2.35. From such curves, one can obtain a linearized formula such as Eq. (2.71) to use in

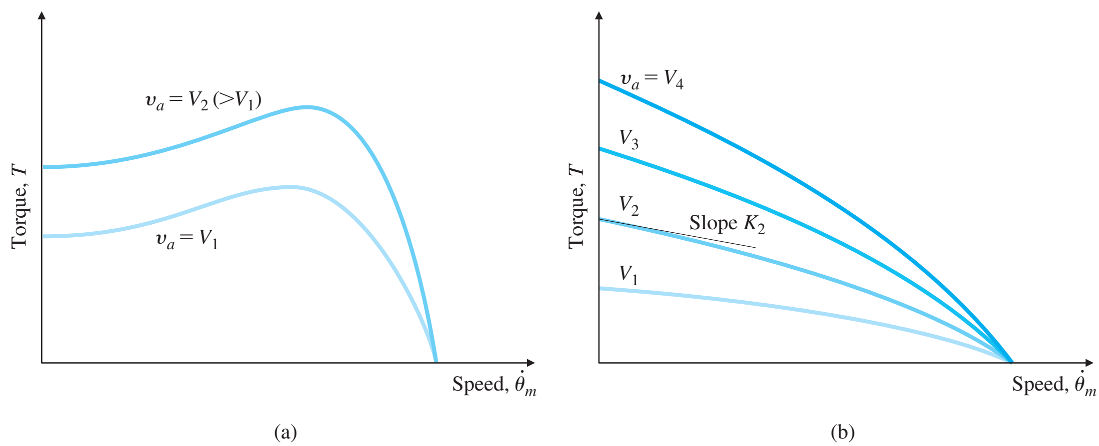


Figure 2.35

Torque-speed curves for a servo motor showing four amplitudes of armature voltage: (a) low-rotor-resistance machine; (b) high-rotor-resistance machine showing four values of armature voltage, v_a

the mechanical part of the system and an equivalent circuit consisting of a resistance and an inductance to use in the electrical part.

△ 2.3.3 Gears

The motors used for control purposes are often used in conjunction with gears as shown in Fig. 2.36 in order to multiply the torque. The force transmitted by the teeth of one gear is equal and opposite to the force applied to the other gear as shown in Fig. 2.36(a); therefore, since torque = force × distance, the torques applied to and from each shaft by the teeth obeys

$$\frac{T_1}{r_1} = \frac{T_2}{r_2} = f, \text{ force applied by teeth} \tag{2.72}$$

and thus, we see that the torque multiplication is proportional to the radius of the gears, r, or equivalently, the number of teeth, N, in each gear,

$$\frac{T_2}{T_1} = \frac{r_2}{r_1} = \frac{N_2}{N_1} = n, \tag{2.73}$$

where we have defined the quantity, n, to be the gear ratio.

Similarly, the velocity of the contact tooth of one gear is the same as the velocity of the tooth on the opposite gear, and since velocity = ωr, where ω is the angular velocity,

$$\omega_1 r_1 = \omega_2 r_2 = v.$$

Thus,

$$\frac{\omega_1}{\omega_2} = \frac{r_2}{r_1} = \frac{N_2}{N_1} = n. \tag{2.74}$$

Furthermore, the angles will change in proportion to the angular velocities, so

$$\frac{\theta_1}{\theta_2} = \frac{\omega_1}{\omega_2} = \frac{N_2}{N_1} = n. \tag{2.75}$$

Note these are all geometric relationships in the sense that we have not considered any inertias or accelerations of the gear train. These relationships simply change the scale factor on the torque and speed from a motor. There is also another effect that must be considered: the

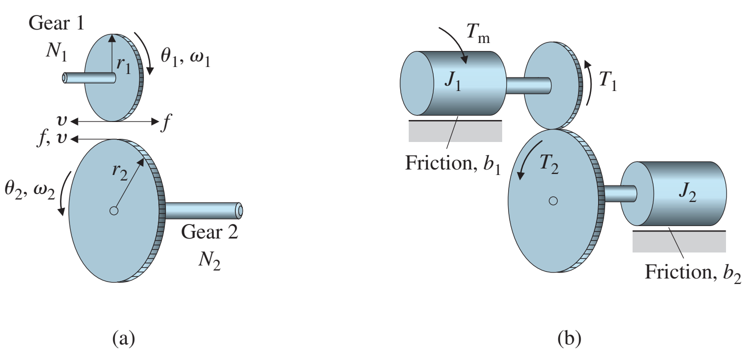


Figure 2.36
(a) Geometry definitions and forces on teeth; (b) definitions for the dynamic analysis

effective rotational inertia and damping of the system when considering the dynamics. Suppose the servo motor whose output torque is T_m is attached to gear 1. Also suppose the servo's gear 1 is meshed with gear 2, and the angle θ_2 describes its position (body 2). Furthermore, the inertia of gear 1 and all that is attached to it (body 1) is J_1, while the inertia of the second gear and all the attached load (body 2) is J_2, similarly for the friction b_1 and b_2. We wish to determine the transfer function between the applied torque, T_m, and the output θ_2, that is, $\Theta_2(s)/T_m(s)$. The equation of motion for body 1 is

$$J_1\ddot{\theta}_1 + b_1\dot{\theta}_1 = T_m - T_1, \tag{2.76}$$

where T_1 is the reaction torque from gear 2 acting back on gear 1. For body 2, the equation of motion is

$$J_2\ddot{\theta}_2 + b_2\dot{\theta}_2 = T_2, \tag{2.77}$$

where T_2 is the torque applied on gear 2 by gear 1. Note that these are not independent systems because the motion is tied together by the gears. Substituting θ_2 for θ_1 in Eq. (2.76) using the relationship from Eq. (2.75), replacing T_2 with T_1 in Eq. (2.77) using the relationship in Eq. (2.73), and eliminating T_1 between the two equations results in

$$(J_2 + J_1 n^2)\ddot{\theta}_2 + (b_2 + b_1 n^2)\dot{\theta}_2 = nT_m. \tag{2.78}$$

So the transfer function is

$$\frac{\Theta_2(s)}{T_m(s)} = \frac{n}{J_{eq}s^2 + b_{eq}s}, \tag{2.79}$$

where

$$J_{eq} = J_2 + J_1 n^2, \text{ and } b_{eq} = b_2 + b_1 n^2. \tag{2.80}$$

These quantities are referred to as the "equivalent" inertias and damping coefficients.[9] If the transfer function had been desired between the applied torque, T_m, and θ_1, a similar analysis would be required to arrive at the equivalent inertias and damping, which would be different from those above.

△ 2.4 Heat and Fluid-Flow Models

Thermodynamics, heat transfer, and fluid dynamics are each the subject of complete textbooks. For purposes of generating dynamic models for use in control systems, the most important aspect of the physics is to represent the dynamic interaction between the variables. Experiments are usually required to determine the actual values of the parameters, and thus to complete the dynamic model for purposes of control systems design.

[9] The equivalent inertia is sometimes referred to as "reflected impedance"; however, this term is more typically applied to electronic circuits.

2.4.1 Heat Flow

Some control systems involve regulation of temperature for portions of the system. The dynamic models of temperature control systems involve the flow and storage of heat energy. Heat energy flows through substances at a rate proportional to the temperature difference across the substance; that is,

$$q = \frac{1}{R}(T_1 - T_2), \tag{2.81}$$

where

q = heat-energy flow, joules per second (J/sec),
R = thermal resistance, °C/J · sec,
T = temperature, °C.

The net heat-energy flow into a substance affects the temperature of the substance according to the relation

$$\dot{T} = \frac{1}{C}q, \tag{2.82}$$

where C is the thermal capacity. Typically, there are several paths for heat to flow into or out of a substance, and q in Eq. (2.82) is the sum of heat flows obeying Eq. (2.81).

EXAMPLE 2.16 *Heat Flow from a Room*

A room with all but two sides insulated ($1/R = 0$) is shown in Fig. 2.37. Find the differential equations that determine the temperature in the room.

Solution. Application of Eqs. (2.81) and (2.82) yields

$$\dot{T}_I = \frac{1}{C_I}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)(T_O - T_I),$$

where

C_I = thermal capacity of air within the room,
T_O = temperature outside,
T_I = temperature inside,
R_2 = thermal resistance of the room ceiling,
R_1 = thermal resistance of the room wall.

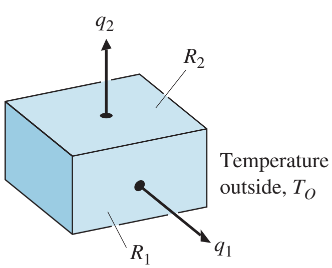


Figure 2.37
Dynamic model for room temperature

Normally the material properties are given in tables as follows:

Specific heat

1. The specific heat at constant volume c_v, which is converted to heat capacity by
$$C = mc_v, \tag{2.83}$$
where m is the mass of the substance;

Thermal conductivity

2. The thermal conductivity[10] k, which is related to thermal resistance R by
$$\frac{1}{R} = \frac{kA}{l},$$
where A is the cross-sectional area and l is the length of the heat-flow path.

EXAMPLE 2.17 *A Thermal Control System*

The system consists of two thermal masses in contact with one another where heat is being applied to the mass on the left, as shown in Fig. 2.38. There is also heat transferred directly to the second mass in contact with it, and heat is lost to the environmnet from both masses. Find the relevant dynamic equations and the transfer function between the heat input, u, and the temperature of the mass on the right.

Solution. Applying Eqs. (2.81) and (2.82) yields
$$C_1\dot{T}_1 = u - H_1T_1 - H_x(T_1 - T_2), \tag{2.84}$$
$$C_2\dot{T}_2 = H_x(T_1 - T_2) - H_2T_2, \tag{2.85}$$
where

C_1 = thermal capacity of mass 1,

C_2 = thermal capacity of mass 2,

T_o = temperature outside the masses,

$T_1 = T_1^* - T_o$ temperature difference of mass 1,

$T_2 = T_2^* - T_o$ temperature difference of mass 2,

$H_1 = 1/R_1$ = thermal resistance from mass 1,

$H_2 = 1/R_2$ = thermal resistance from mass 2,

$H_x = 1/R_x$ = thermal resistance from mass 1 to mass 2.

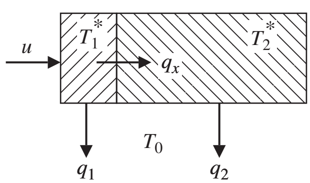


Figure 2.38 A Thermal Control System

[10] In the case of insulation for houses, resistance is quoted as R-values; for example, R-11 refers to a substance that has a resistance to heat-flow equivalent to that given by 11 in. of solid wood.

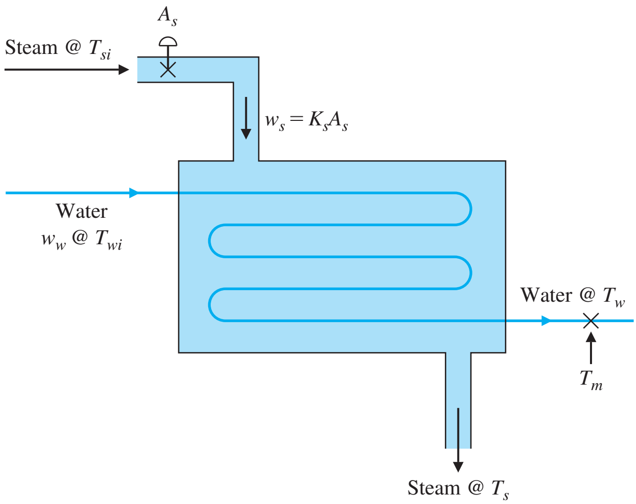


Figure 2.39
Heat exchanger

Using Cramer's Rule with Eqs. (2.84) and (2.85) yields the desired transfer function

$$\frac{T_2(s)}{U(s)} = \frac{H_x}{(C_1 s + H_x + H_1)(C_2 s + H_x + H_2)}. \tag{2.86}$$

In addition to flow due to transfer, as expressed by Eq. (2.81), heat can also flow when a warmer mass flows into a cooler mass, or vice versa. In this case,

$$q = wc_v(T_1 - T_2), \tag{2.87}$$

where w is the mass flow rate of the fluid at T_1 flowing into the reservoir at T_2. For a more complete discussion of dynamic models for temperature control systems, see Cannon (1967) or textbooks on heat transfer.

EXAMPLE 2.18 *Equations for Modeling a Heat Exchanger*

A heat exchanger is shown in Fig. 2.39. Steam enters the chamber through the controllable valve at the top, and cooler steam leaves at the bottom. There is a constant flow of water through the pipe that winds through the middle of the chamber so it picks up heat from the steam. Find the differential equations that describe the dynamics of the measured water outflow temperature as a function of the area A_s of the steam-inlet control valve when open. The sensor that measures the water outflow temperature, being downstream from the exit temperature in the pipe, lags the temperature by t_d sec.

Solution. The temperature of the water in the pipe will vary continuously along the pipe as the heat flows from the steam to the water. The temperature of the steam will also reduce in the chamber as it passes over the maze of pipes. An accurate thermal model of this process is therefore quite involved because the actual heat transfer from the steam to the water will be proportional to the local temperatures of each fluid. For many control applications, it is not necessary to have great accuracy because the feedback will correct for a considerable amount of error in the model. Therefore, it makes sense to combine the spatially varying temperatures into single temperatures T_s and T_w for the outflow steam and water temperatures, respectively. We then assume the heat transfer from steam to water is proportional to the difference in these temperatures, as given by Eq. (2.81). There is also a flow of heat into the chamber from the inlet steam that depends on the steam flow rate and its temperature according to Eq. (2.87),

$$q_{in} = w_s c_{vs}(T_{si} - T_s),$$

where

$w_s = K_s A_s$, mass flow rate of the steam,

$A_s =$ area of the steam inlet valve,

$K_s =$ flow coefficient of the inlet valve,

$c_{vs} =$ specific heat of the steam,

$T_{si} =$ temperature of the inflow steam,

$T_s =$ temperature of the outflow steam.

The net heat flow into the chamber is the difference between the heat from the hot incoming steam and the heat flowing out to the water. This net flow determines the rate of temperature change of the steam according to Eq. (2.82),

$$C_s \dot{T}_s = A_s K_s c_{vs}(T_{si} - T_s) - \frac{1}{R}(T_s - T_w), \tag{2.88}$$

where

$C_s = m_s c_{vs}$ is the thermal capacity of the steam in the chamber with mass m_s,

$R =$ the thermal resistance of the heat flow averaged over the entire exchanger.

Likewise, the differential equation describing the water temperature is

$$C_w \dot{T}_w = w_w c_{cw}(T_{wi} - T_w) + \frac{1}{R}(T_s - T_w), \tag{2.89}$$

where

$w_w =$ mass flow rate of the water,

$c_{cw} =$ specific heat of the water,

$T_{wi} =$ temperature of the incoming water,

$T_w =$ temperature of the outflowing water.

To complete the dynamics, the time delay between the measurement and the exit flow is described by the relation

$$T_m(t) = T_w(t - t_d),$$

where T_m is the measured downstream temperature of the water and t_d is the time delay. There may also be a delay in the measurement of the steam temperature T_s, which would be modeled in the same manner.

Equation (2.88) is nonlinear because the quantity T_s is multiplied by the control input A_s. The equation can be linearized about T_{so} (a specific value of T_s) so $T_{si} - T_s$ is assumed constant for purposes of approximating the nonlinear term, which we will define as ΔT_s. In order to eliminate the T_{wi} term in Eq. (2.89), it is convenient to measure all temperatures in terms of deviation in degrees from T_{wi}. The resulting equations are then

$$\begin{aligned} C_s\dot{T}_s &= -\frac{1}{R}T_s + \frac{1}{R}T_w + K_s c_{vs}\Delta T_s A_s, \\ C_w\dot{T}_w &= -\left(\frac{1}{R} + w_w c_{vw}\right)T_w + \frac{1}{R}T_s, \\ T_m &= T_w(t - t_d). \end{aligned}$$

Although the time delay is not a nonlinearity, we will see in Chapter 3 that operationally, $T_m = e^{-t_d s}T_w$. Therefore, the transfer function of the heat exchanger has the form

$$\frac{T_m(s)}{A_s(s)} = \frac{Ke^{-t_d s}}{(\tau_1 s + 1)(\tau_2 s + 1)}. \tag{2.90}$$

2.4.2 Incompressible Fluid Flow

Hydraulic actuator

Fluid flows are common in many control system components. One example is the hydraulic actuator, which is used extensively in control systems because it can supply a large force with low inertia and low weight. They are often used to move the aerodynamic control surfaces of airplanes; to gimbal rocket nozzles; to move the linkages in earth-moving equipment, farm tractor implements, snow-grooming machines; and to move robot arms.

The continuity relation

The physical relations governing fluid flow are continuity, force equilibrium, and resistance. The continuity relation is simply a statement of the conservation of matter; that is,

$$\dot{m} = w_{in} - w_{out}, \tag{2.91}$$

where

$$\begin{aligned} m &= \text{fluid mass within a prescribed portion of the system,} \\ w_{in} &= \text{mass flow rate into the prescribed portion of the system,} \\ w_{out} &= \text{mass flow rate out of the prescribed portion of the system.} \end{aligned}$$

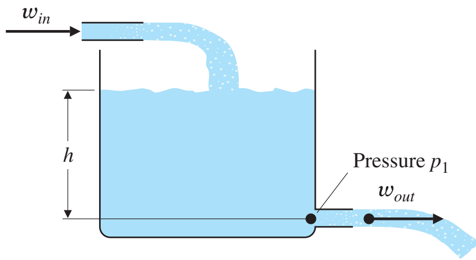


Figure 2.40
Water tank example

EXAMPLE 2.19

Equations for Describing Water Tank Height

Determine the differential equation describing the height of the water in the tank in Fig. 2.40.

Solution. Application of Eq. (2.91) yields

$$\dot{h} = \frac{1}{A\rho}(w_{in} - w_{out}), \tag{2.92}$$

where

$A =$ area of the tank,

$\rho =$ density of water,

$h = m/A\rho =$ height of water,

$m =$ mass of water in the tank.

Force equilibrium must apply exactly as described by Eq. (2.1) for mechanical systems. Sometimes in fluid-flow systems, some forces result from fluid pressure acting on a piston. In this case, the force from the fluid is

$$f = pA, \tag{2.93}$$

where

$f =$ force,

$p =$ pressure in the fluid,

$A =$ area on which the fluid acts.

EXAMPLE 2.20

Modeling a Hydraulic Piston

Determine the differential equation describing the motion of the piston actuator shown in Fig. 2.41, given that there is a force F_D acting on it and a pressure p in the chamber.

Solution. Equations (2.1) and (2.93) apply directly, where the forces include the fluid pressure as well as the applied force. The result is

$$M\ddot{x} = Ap - F_D,$$

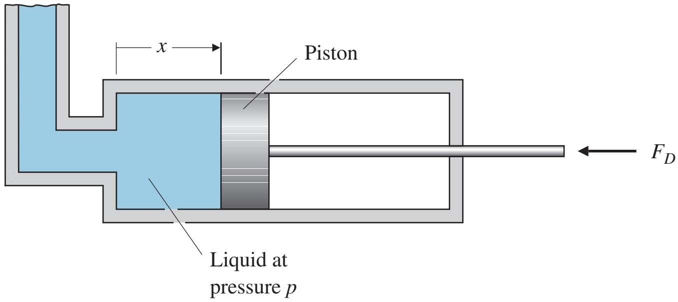


Figure 2.41
Hydraulic piston actuator

where

$$A = \text{area of the piston,}$$
$$p = \text{pressure in the chamber,}$$
$$M = \text{mass of the piston,}$$
$$x = \text{position of the piston.}$$

In many cases of fluid-flow problems, the flow is resisted either by a constriction in the path or by friction. The general form of the effect of resistance is given by

$$w = \frac{1}{R}(p_1 - p_2)^{1/\alpha}, \tag{2.94}$$

where

w = mass flow rate,

p_1, p_2 = pressures at ends of the path through which flow is occurring,

R, α = constants whose values depend on the type of restriction.

Or, as is more commonly used in hydraulics,

$$Q = \frac{1}{\rho R}(p_1 - p_2)^{1/\alpha}, \tag{2.95}$$

where

Q = volume flow rate, where $Q = w/\rho$,

ρ = fluid density.

The constant α takes on values between 1 and 2. The most common value is approximately 2 for high flow rates (those having a Reynolds number $\text{Re} > 10^5$) through pipes or through short constrictions or nozzles. For very slow flows through long pipes or porous plugs wherein the flow remains laminar ($\text{Re} \lesssim 1000$), $\alpha = 1$. Flow rates between these extremes can yield intermediate values of α. The Reynolds number indicates the relative importance of inertial forces and viscous forces in the flow. It is proportional to a material's velocity and density and to the size of the restriction, and it is inversely proportional to the viscosity. When Re is small, the viscous forces predominate and the flow is laminar. When Re is large, the inertial forces predominate and the flow is turbulent.

Note a value of $\alpha = 2$ indicates that the flow is proportional to the square root of the pressure difference and therefore will produce a nonlinear differential equation. For the initial stages of control systems analysis and design, it is typically very useful to linearize these equations so the design techniques described in this book can be applied. Linearization involves selecting an operating point and expanding the nonlinear term to be a small perturbation from that point.

EXAMPLE 2.21 *Linearization of Water Tank Height and Outflow*

Find the nonlinear differential equation describing the height of the water in the tank in Fig. 2.40. Assume there is a relatively short restriction at the outlet and that $\alpha = 2$. Also linearize your equation about the operating point h_o.

Solution. Applying Eq. (2.94) yields the flow out of the tank as a function of the height of the water in the tank:

$$w_{out} = \frac{1}{R}(p_1 - p_a)^{1/2}. \tag{2.96}$$

Here,

$p_1 = \rho g h + p_a$, the hydrostatic pressure,

$p_a =$ ambient pressure outside the restriction.

Substituting Eq. (2.96) into Eq. (2.92) yields the nonlinear differential equation for the height:

$$\dot{h} = \frac{1}{A\rho}\left(w_{in} - \frac{1}{R}\sqrt{p_1 - p_a}\right). \tag{2.97}$$

Linearization involves selecting the operating point $p_o = \rho g h_o + p_a$ and substituting $p_1 = p_o + \Delta p$ into Eq. (2.96). Then, we expand the nonlinear term according to the relation

$$(1+\varepsilon)^{\beta} \cong 1 + \beta\varepsilon, \tag{2.98}$$

where $\varepsilon \ll 1$. Equation (2.96) can thus be written as

$$\begin{aligned} w_{out} &= \frac{\sqrt{p_o - p_a}}{R}\left(1 + \frac{\Delta p}{p_o - p_a}\right)^{1/2} \\ &\cong \frac{\sqrt{p_o - p_a}}{R}\left(1 + \frac{1}{2}\frac{\Delta p}{p_o - p_a}\right). \end{aligned} \tag{2.99}$$

The linearizing approximation made in Eq. (2.99) is valid as long as $\Delta p \ll p_o - p_a$; that is, as long as the deviations of the system pressure from the chosen operating point are relatively small.

Combining Eqs. (2.92) and (2.99) yields the following linearized equation of motion for the water tank level:

$$\Delta\dot{h} = \frac{1}{A\rho}\left[w_{in} - \frac{\sqrt{p_o - p_a}}{R}\left(1 + \frac{1}{2}\frac{\Delta p}{p_o - p_a}\right)\right].$$

Because $\Delta p = \rho g \Delta h$, this equation reduces to

$$\Delta\dot{h} = -\frac{g}{2AR\sqrt{p_o - p_a}}\Delta h + \frac{w_{in}}{A\rho} - \frac{\sqrt{p_o - p_a}}{\rho AR}, \qquad (2.100)$$

which is a linear differential equation for $\Delta\dot{h}$. The operating point is not an equilibrium point because some control input is required to maintain it. In other words, when the system is at the operating point ($\Delta h = 0$) with no input ($w_{in} = 0$), it will move from that point because $\Delta\dot{h} \neq 0$. So, if no water is flowing into the tank, the tank will drain, thus moving it from the reference point. To define an operating point that is also an equilibrium point, we need to require that there be a nominal flow rate,

$$\frac{w_{in_o}}{A\rho} = \frac{\sqrt{p_o - p_a}}{\rho AR},$$

and define the linearized input flow to be a perturbation from that value.

Hydraulic actuators

Hydraulic actuators obey the same fundamental relationships we saw in the water tank: continuity [Eq. (2.91)], force balance [Eq. (2.93)], and flow resistance [Eq. (2.94)]. Although the development here assumes the fluid to be perfectly incompressible, in fact, hydraulic fluid has some compressibility due primarily to entrained air. This feature causes hydraulic actuators to have some resonance because the compressibility of the fluid acts like a stiff spring. This resonance limits their speed of response.

EXAMPLE 2.22 *Modeling a Hydraulic Actuator*

1. Find the nonlinear differential equations relating the movement θ of the control surface to the input displacement x of the valve for the hydraulic actuator shown in Fig. 2.42.
2. Find the linear approximation to the equations of motion when $\dot{y}$ = constant, with and without an applied load—that is, when $F \neq 0$ and when $F = 0$. Assume θ motion is small.

Solution

1. **Equations of motion:** When the valve is at $x = 0$, both passages are closed and no motion results. When $x > 0$, as shown in Fig. 2.42, the oil flows clockwise as shown and the piston is forced to the left. When $x < 0$, the fluid flows counterclockwise. The oil supply at high pressure p_s enters the *left* side of the large piston chamber, forcing the piston to the right. This causes the oil to flow out of the valve chamber from the rightmost channel instead of the left.

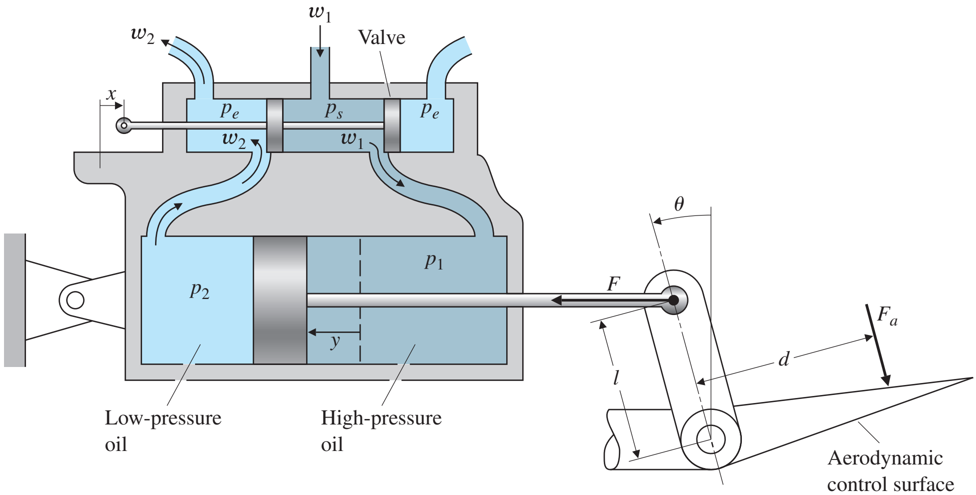


Figure 2.42
Hydraulic actuator with valve

We assume the flow through the orifice formed by the valve is proportional to x; that is,

$$Q_1 = \frac{1}{\rho R_1}(p_s - p_1)^{1/2}x. \tag{2.101}$$

Similarly,

$$Q_2 = \frac{1}{\rho R_2}(p_2 - p_e)^{1/2}x. \tag{2.102}$$

The continuity relation yields

$$A\dot{y} = Q_1 = Q_2, \tag{2.103}$$

where

$$A = piston\ area.$$

The force balance on the piston yields

$$A(p_1 - p_2) - F = m\ddot{y}, \tag{2.104}$$

where

m = mass of the piston and the attached rod,

F = force applied by the piston rod to the control surface attachment point.

Furthermore, the moment balance of the control surface using Eq. (2.10) yields

$$I\ddot{\theta} = Fl\cos\theta - F_a d, \tag{2.105}$$

where

I = moment of inertia of the control surface and attachment about the hinge,

F_a = applied aerodynamic load.

To solve this set of five equations, we require the following additional kinematic relationship between θ and y:

$$y = l\sin\theta. \tag{2.106}$$

The actuator is usually constructed so the valve exposes the two passages equally; therefore, $R_1 = R_2$, and we can infer from Eqs. (2.101) to (2.103) that

$$p_s - p_1 = p_2 - p_e. \tag{2.107}$$

These relations complete the nonlinear differential equations of motion; they are formidable and difficult to solve.

2. **Linearization and simplification:** For the case in which $\dot{y}$ = a constant ($\ddot{y} = 0$) and there is no applied load ($F = 0$), Eqs. (2.104) and (2.107) indicate that

$$p_1 = p_2 = \frac{p_s + p_e}{2}. \tag{2.108}$$

Therefore, using Eq. (2.103) and letting $\sin\theta = \theta$ (since θ is assumed to be small), we get

$$\dot{\theta} = \frac{\sqrt{p_s - p_e}}{\sqrt{2}A\rho Rl}x. \tag{2.109}$$

This represents a single integration between the input x and the output θ, where the proportionality constant is a function only of the supply pressure and the fixed parameters of the actuator. For the case $\dot{y}$ = constant but $F \neq 0$, Eqs. (2.104) and (2.107) indicate that

$$p_1 = \frac{p_s + p_e + F/A}{2}$$

and

$$\dot{\theta} = \frac{\sqrt{p_s - p_e - F/A}}{\sqrt{2}A_\rho Rl}x. \tag{2.110}$$

This result is also a single integration between the input x and the output θ, but the proportionality constant now depends on the applied load F.

As long as the commanded values of x produce θ motion that has a sufficiently small value of $\ddot{\theta}$, the approximation given by Eq. (2.109) or (2.110) is valid and no other linearized dynamic relationships are necessary. However, as soon as the commanded values of x produce accelerations in which the inertial forces ($m\ddot{y}$ and the reaction to $I\ddot{\theta}$) are a significant fraction of $p_s - p_e$, the approximations are no longer valid. We must then incorporate these forces into the equations, thus obtaining a dynamic relationship between x and θ that is much more involved than the pure integration implied by Eq. (2.109) or (2.110). Typically, for initial control system designs, hydraulic actuators are assumed to obey the simple relationship of Eq. (2.109) or (2.110). When hydraulic actuators are used in feedback control systems, resonances have been encountered that are not explained by using the approximation that the device is a simple integrator as in Eq. (2.109) or (2.110). The source of the resonance is the neglected accelerations discussed above along with the additional feature that the oil is slightly compressible due to small quantities of entrained air. This phenomenon is called the “oil-mass resonance.”

2.5 Historical Perspective

Newton’s second law of motion (Eq. 2.1) was first published in his *Philosophiæ Naturalis Principia Mathematica* in 1686 along with his two other famous laws of motion. The first: A body will continue with the same uniform motion unless acted on by an external unbalanced force, and the third: To every action, there is an equal and opposite reaction. Isaac Newton also published his law of gravitation in this same publication, which stated that every mass particle attracts all other particles by a force proportional to the inverse of the square of the distance between them and the product of their two masses. His basis for developing these laws was the work of several other early scientists, combined with his own development of the calculus in order to reconcile all the observations. It is amazing that these laws still stand today as the basis for almost all dynamic analysis with the exception of Einstein’s additions in the early 1900s for relativistic effects. It is also amazing that Newton’s development of calculus formed the foundation of our mathematics that enable dynamic modeling. In addition to being brilliant, he was also very eccentric. As Brennan writes in *Heisenberg Probably Slept Here*, “He was seen about campus in his disheveled clothes, his wig askew, wearing run-down shoes and a soiled neckpiece. He seemed to care about nothing but his work. He was so absorbed in his studies that he forgot to eat.” Another interesting aspect of Newton is that he initially developed the calculus and the now famous laws of physics about 20 years prior to publishing them! The incentive to publish them arose from a bet between three men having lunch at a pub in 1684: Edmond Halley, Christopher Wren, and Robert Hooke. They all had the opinion that Kepler’s elliptical characterization of planetary motion could be explained by the inverse square law, but nobody had ever proved it, so they “placed a bet as to who could first prove the conjecture.”[11] Halley went to Newton for help due to his fame as a mathematician, who responded he had already done it many years ago and would forward the papers to him. He not only did that shortly afterward, but followed it up with the *Principia* with all the details two years later.

[11] Much of the background on Newton was taken from *Heisenberg Probably Slept Here*, by Richard P. Brennan, 1997. The book discusses his work and the other early scientists that laid the groundwork for Newton.

The basis for Newton's work started with the astronomer Nicholas Copernicus more than a hundred years before the *Principia* was published. He was the first to speculate that the planets revolved around the sun, rather than everything in the skies revolving around the earth. But Copernicus' heretical notion was largely ignored at the time, except by the church who banned his publication. However, two scientists did take note of his work: Galileo Galilei in Italy and Johannes Kepler in Austria. Kepler relied on a large collection of astronomical data taken by a Danish astronomer, Tycho Brahe, and concluded that the planetary orbits were ellipses rather than the circles that Copernicus had postulated. Galileo was an expert telescope builder and was able to clearly establish that the earth was not the center of all motion, partly because he was able to see moons revolving around other planets. He also did experiments with rolling balls down inclined planes that strongly suggested that $F = ma$ (alas, it's a myth that he did his experiments by dropping objects out of the Leaning Tower of Pisa). Galileo published his work in 1632, which raised the ire of the church who then later banned him to house arrest until he died.[12] It was not until 1985 that the church recognized the important contributions of Galileo! These men laid the groundwork for Newton to put it all together with his laws of motion and the inverse square gravitational law. With these two physical principles, all the observations fit together with a theoretical framework that today forms the basis for the modeling of dynamic systems.

The sequence of discoveries that ultimately led to the laws of dynamics that we take for granted today were especially remarkable when we stop to think that they were all carried out without a computer, a calculator, or even a slide rule. On top of that, Newton had to invent calculus in order to reconcile the data.

After publishing the *Principia*, Newton went on to be elected to Parliament and was given high honors, including being the first man of science to be knighted by the Queen. He also got into fights with other scientists fairly regularly and used his powerful positions to get what he wanted. In one instance, he wanted data from the Royal Observatory that was not forthcoming fast enough. So he created a new board with authority over the Observatory and had the Astronomer Royal expelled from the Royal Society. Newton also had other less scientific interests. Many years after his death, John Maynard Keynes found that Newton had been spending as much of his time on metaphysical occult, alchemy, and biblical works as he had been on physics.

More than a hundred years after Newton's *Principia*, Michael Faraday performed a multitude of experiments and postulated the notion of electromagnetic lines of force in free space. He also discovered induction (Faraday's Law), which led to the electric motor and the laws of electrolysis. Faraday was born into a poor family, had virtually no schooling, and became an apprentice to a bookbinder at age 14. There

[12]Galileo's life, accomplishments, and house arrest are very well described in Dava Sobel's book, *Galileo's Daughter.*

he read many of the books being bound and became fascinated by science articles. Enthralled by these, he maneuvered to get a job as a bottle washer for a famous scientist, eventually learned enough to be a competitor to him, and ultimately became a professor at the Royal Institution in London. But lacking a formal education, he had no mathematical skills, and lacked the ability to create a theoretical framework for his discoveries. Faraday became a famous scientist in spite of his humble origins. After he had achieved fame for his discoveries and was made a Fellow of the Royal Society, the prime minister asked him what good his inventions could be.[13] Faraday's answer was, "Why Prime Minister, someday you can tax it." But in those days, scientists were almost exclusively men born into privilege; so Faraday had been treated like a second-class citizen by some of the other scientists. As a result, he rejected knighthood as well as burial at Westminster Abbey. Faraday's observations, along with those by Coulomb and Ampere, led James Clerk Maxwell to integrate all their knowledge on magnetism and electricity into Maxwell's equations. Against the beliefs of most prominent scientists of the day (Faraday being an exception), Maxwell invented the concepts of fields and waves that explained magnetic and electrostatic forces and was the key to creating the unifying theory. Although Newton had discovered the spectrum of light, Maxwell was also the first to realize that light was one type of the same electromagnetic waves, and its behavior was explained as well by Maxwell's equations. In fact, the only constants in his equations are μ and ε. The constant speed of light is $c = 1/\sqrt{\mu\varepsilon}$.

Maxwell was a Scottish mathematician and theoretical physicist. His work has been called the second great unification in physics, the first being that due to Newton. Maxwell was born into the privileged class and was given the benefits of an excellent education and excelled at it. In fact, he was an extremely gifted theoretical and experimental scientist as well as a very generous and kind man with many friends and little vanity. In addition to unifying the observations of electromagnetics into a theory that still governs our engineering analyses today, he was the first to present an explanation of how light travels, the primary colors, the kinetic theory of gases, the stability of Saturn's rings, and the stability of feedback control systems! His discovery of the three primary colors (red, green, and blue) forms the basis of our color television to this day. His theory showing the speed of light is a constant was difficult to reconcile with Newton's laws and led Albert Einstein to create the special theory of relativity in the early 1900s. This led Einstein to say, "One scientific epoch ended and another began with James Clerk Maxwell."[14]

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[13] *$E = MC^2$, A Biography of the World's Most Famous Equation*, by David Bodanis, Walker and Co., New York, 2000.

[14] *The Man Who Changed Everything: The Life of James Clerk Maxwell*, Basil Mahon, Wiley, Chichester, UK, 2003.

SUMMARY

Mathematical modeling of the system to be controlled is the first step in analyzing and designing the required system controls. In this chapter we developed analytical models for representative systems. Important equations for each category of system are summarized in Table 2.1. It is also possible to obtain a mathematical model using experimental data exclusively. This approach will be discussed briefly in Chapter 3 and more extensively in Chapter 12 of Franklin, Powell, and Workman (1998).

TABLE 2.1 **Key Equations for Dynamic Models**

System	Important Laws or Relationships	Associated Equations	Equation Number(s)
Mechanical	Translational motion (Newton's law)	$F = ma$	(2.1)
	Rotational motion	$M = I\alpha$	(2.10)
Electrical	Operational amplifier		(2.46), (2.47)
Electromechanical	Law of motors	$F = Bli$	(2.53)
	Law of generators	$e = Blv$	(2.56)
	Torque developed in a rotor	$T = K_t i_a$	(2.60)
Back emf	Voltage generated as a result of rotation of a rotor	$e = K_e \dot{\theta}_m$	(2.61)
Gears	Effective inertia	$J_{eq} = J_2 + J_1 n^2$	(2.80)
Heat flow	Heat-energy flow	$q = {}^1\!/_R(T_1 - T_2)$	(2.81)
	Temperature as a function of heat-energy flow	$\dot{T} = \frac{1}{C} q$	(2.82)
	Specific heat	$C = mc_v$	(2.83)
Fluid flow	Continuity relation (conservation of matter)	$\dot{m} = w_{in} - w_{out}$	(2.91)
	Force of a fluid acting on a piston	$f = pA$	(2.93)
	Effect of resistance to fluid flow	$w = {}^1\!/_R(p_1 - p_2)^{1/\alpha}$	(2.94)

REVIEW QUESTIONS

2.1 What is a "free-body diagram"?

2.2 What are the two forms of Newton's law?

2.3 For a structural process to be controlled, such as a robot arm, what is the meaning of "collocated control"? "Noncollocated control"?

2.4 State Kirchhoff's current law.

2.5 State Kirchhoff's voltage law.

2.6 When, why, and by whom was the device named an "operational amplifier"?

2.7 What is the major benefit of having zero input current to an operational amplifier?

2.8 Why is it important to have a small value for the armature resistance R_a of an electric motor?

2.9 What are the definition and units of the electric constant of a motor?

2.10 What are the definition and units of the torque constant of an electric motor?

2.11 Why do we approximate a physical model of the plant (which is *always* nonlinear) with a linear model?

△ **2.12** Give the relationships for the following:

(a) Heat flow across a substance

(b) Heat storage in a substance

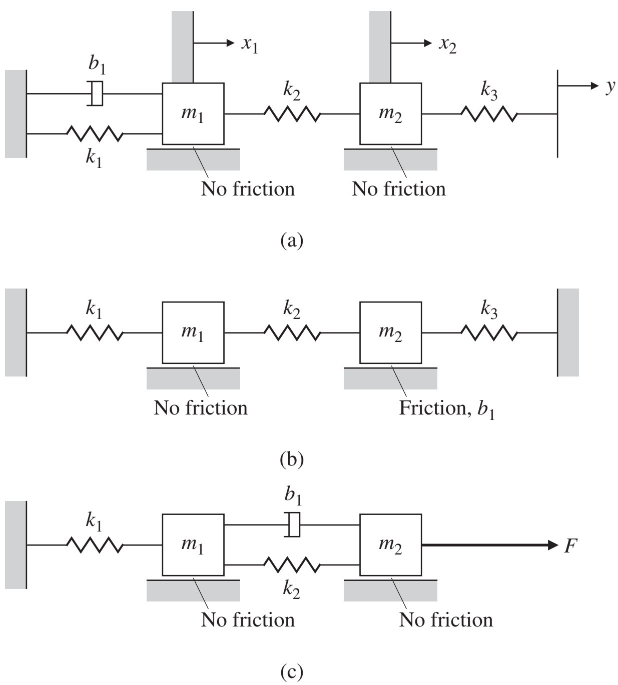
△ **2.13** Name and give the equations for the three relationships governing fluid flow.

PROBLEMS

Problems for Section 2.1: Dynamics of Mechanical Systems

2.1 Write the differential equations for the mechanical systems shown in Fig. 2.43. For Fig. 2.43(a) and (b), state whether you think the system will eventually decay so it has no motion at all, given that there are nonzero

Figure 2.43
Mechanical systems



initial conditions for both masses and there is no input; give a reason for your answer. Also, for part (c), answer the question for $F = 0$.

2.2 Write the differential equation for the mechanical system shown in Fig. 2.44. State whether you think the system will eventually decay so it has no motion at all, given that there are nonzero initial conditions for both masses, and give a reason for your answer.

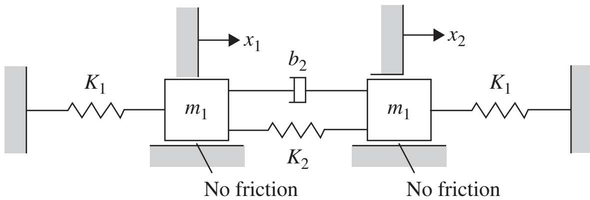


Figure 2.44
Mechanical system for Problem 2.2

2.3 Write the equations of motion for the double-pendulum system shown in Fig. 2.45. Assume the displacement angles of the pendulums are small enough to ensure the spring is always horizontal. The pendulum rods are taken to be massless, of length l, and the springs are attached three-fourths of the way down.

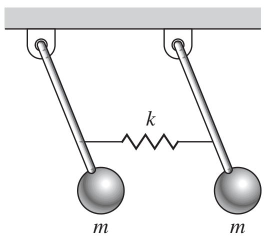


Figure 2.45
Double pendulum

2.4 Write the equations of motion of a pendulum consisting of a thin, 2 kg stick of length l suspended from a pivot. How long should the rod be in order for the period to be exactly 1 sec? (The inertia I of a thin stick about an end point is $\frac{1}{3}ml^2$. Assume θ is small enough that $\sin\theta \cong \theta$.) Why do you think grandfather clocks are typically about 6 ft high?

2.5 For the car suspension discussed in Example 2.2, plot the position of the car and the wheel after the car hits a "unit bump"(that is, r is a unit step) using Matlab. Assume $m_1 = 10$ kg, $m_2 = 350$ kg, $K_w = 500{,}000$ N/m, and $K_s = 10{,}000$ N/m. Find the value of b that you would prefer if you were a passenger in the car.

2.6 For the quadcopter shown in Figs. 2.13 and 2.14:

(a) Determine the appropriate commands to rotor #s 1, 2, 3, & 4 so a pure vertical force will be applied to the quadcopter, that is, a force that will have no effect on pitch, roll, or yaw.

(b) Determine the transfer function between F_h, and altitude, h. That is, find $h(s)/F_h(s)$.

2.7 Automobile manufacturers are contemplating building active suspension systems. The simplest change is to make shock absorbers with a changeable damping, $b(u_1)$. It is also possible to make a device to be placed in parallel with the springs that has the ability to supply an equal force, u_2, in opposite directions on the wheel axle and the car body.

(a) Modify the equations of motion in Example 2.2 to include such control inputs.

(b) Is the resulting system linear?

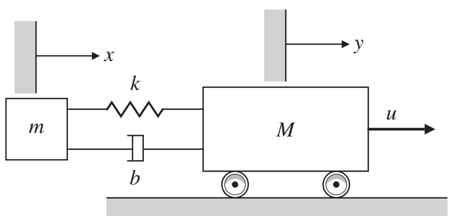
(c) Is it possible to use the force u_2 to completely replace the springs and shock absorber? Is this a good idea?

2.8 In many mechanical positioning systems, there is flexibility between one part of the system and another. An example is shown in Fig. 2.7 where there is flexibility of the solar panels. Figure 2.46 depicts such a situation, where a force u is applied to the mass M and another mass m is connected to it. The coupling between the objects is often modeled by a spring constant k with a damping coefficient b, although the actual situation is usually much more complicated than this.

(a) Write the equations of motion governing this system.

(b) Find the transfer function between the control input u and the output y.

Figure 2.46
Schematic of a system with flexibility



2.9 Modify the equation of motion for the cruise control in Example 2.1, Eq. (2.4), so it has a control law; that is, let

$$u = K(v_r - v), \tag{2.111}$$

where

$$v_r = \text{reference speed}, \tag{2.112}$$

$$K = \text{constant}. \tag{2.113}$$

This is a "proportional"control law in which the difference between v_r and the actual speed is used as a signal to speed the engine up or slow it down. Revise the equations of motion with v_r as the input and v as the output and find the transfer function. Assume $m = 1500$ kg and $b = 70$ N·sec/m, and find the response for a unit step in v_r using Matlab. Using trial and error, find a value of K that you think would result in a control system in which the actual speed converges as quickly as possible to the reference speed with no objectionable behavior.

2.10 Determine the dynamic equations for lateral motion of the robot in Fig. 2.47. Assume it has three wheels with a single, steerable wheel in the front where the controller has direct control of the rate of change of the steering angle, U_{steer}, with geometry as shown in Fig. 2.48. Assume the robot is going in approximately a straight line and its angular deviation from that straight line is very small. Also assume the robot is traveling at a constant speed, V_o. The dynamic equations relating the lateral velocity of the center of the robot as a result of commands in U_{steer} are desired.

Figure 2.47
Robot for delivery of hospital supplies
Source: Bill Clark/Daily Progress/AP Images

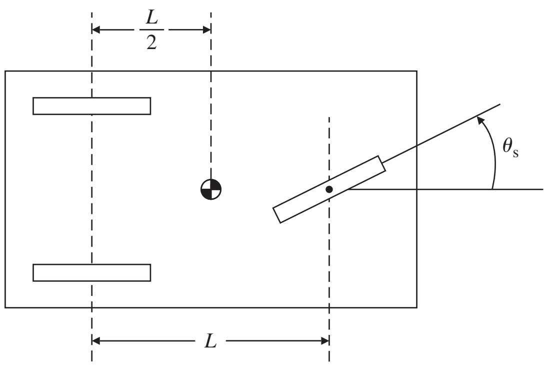


Figure 2.48
Model for robot motion

2.11 Determine the pitch, yaw, and roll control equations for the hexacopter shown in Fig. 2.49 that are similar to those for the quadcopter given in Eqs. (2.18) to (2.20).

Assume rotor #1 is in the direction of flight, and the remaining rotors are numbered CW from that rotor. In other words, rotors #1 and #4 will determine the pitch motion. Rotor #s 2, 3, 5, & 6 will determine roll motion. Pitch, roll and yaw motions are defined by the coordinate system shown in Fig. 2.14 in Example 2.5. In addition to developing the equations for the 3 degrees of freedom in terms of how the six rotor motors should be commanded (similar to those for the quadrotor in Eqs. (2.18)–(2.20)), it will also be necessary to decide which rotors are

Figure 2.49
Hexacopter

turning CW and which ones are turning CCW. The direction of rotation for the rotors needs to be selected so there is no net torque about the vertical axis; that is, the hexicopter will have no tendancy for yaw rotation in steady-state. Furthermore, a control action to affect pitch should have no effect on yaw or roll. Likewise, a control action for roll should have no effect on pitch or yaw, and a control action for yaw should have no effect on pitch or roll. In other words, the control actions should produce no cross-coupling between pitch, roll, and yaw just as was the case for the quadcopter in Example 2.5.

2.12 In most cases, quadcopters have a camera mounted that does not swivel in the x-y plane and its direction of view is oriented at 45° to the arms supporting the rotors. Therefore, these drones typically fly in a direction that is aligned with the camera rather than along an axis containing two of the rotors. To simplify the flight dynamics, the x-direction of the coordinate system is aligned with the camera direction. Based on the coordinate definitions for the axes in Fig. 2.14, assume the x-axis lies half way between rotors # 1 and 2 and determine the rotor commands for the four rotors that would accomplish independent motion for pitch, roll, and yaw.

Problems for Section 2.2: Models of Electric Circuits

2.13 A first step toward a realistic model of an op-amp is given by the following equations and is shown in Fig. 2.50:

$$V_{out} = \frac{10^7}{s+1}[v_+ - v_-],$$
$$i_+ = i_- = 0.$$

Find the transfer function of the simple amplification circuit shown using this model.

Figure 2.50
Circuit for Problem 2.13