Granada Lectures on Effective Field Theories
(J.Santiago's VERY PRELIMINARY version)

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Abstract

This is a practical introduction to effective field theories applied to particle physics.

*This is work in progress, please feel free to send any comments/corrections to jsantiago@ugr.es

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1 Motivation and Introduction

1.1 Why effective field theory?

The main reason to use effective field theories (EFTs) is that Nature decouples. Observations always have a finite precision. Given this precision, all details of the theory that produce smaller effects are irrelevant. Quantum (field) theories make this statement more subtle, as we will see, because virtual effects probe all scales. Still, the effective field theory formalism we are going to dicuss in these lectures provide well-defined methods to encode the effects of short distance (high energies) physics at larger distances (smaller energies). There are also other reasons to use EFTs in particle physics:

- Because it simpler. EFTs provide methods to turn complicated, multi-scale problems into a series of simpler, single-scale ones.
- Because (sometimes) we have to (I). In some cases we do not know the details of the theory at smaller distances, or we do not know how to compute with it (because it is strongly coupled, for example). In this case we can use EFTs as a general and convenient parametrization of *any* short scale physics.
- Because (sometimes) we have to (II). The presence of large logarithms in physical problems with wildly separated scales can break perturbation theory, even in renormalisable models. EFTs provide methods to resum these large logarithms, effectively reorganising the perturbative expansion into a better behaved one.

EFTs have applications well beyond particle physics. However in these lectures we will mostly focus on particle physics applications and will have in general a relativistic quantum field theory (QFT) in mind. Thus, short distances correspond to high energies and long distances to small energies. We will use both indistinctively and will talk about ultraviolet (UV) physics when dealing with high-energy/short-distance effects and infrared (IR) physics when dealing with low-energy/long-distance effects. Also in general, for the sake of concreteness, we will usually have in mind some heavy physics, with a mass scale much larger than the energies that can be experimentally probed, and we want to compute their effects on physical observables at such energies. We will normally call the theory that includes the heavy particles the full theory and the theory that parametrises their effect at low energies simply the EFT or IR theory.

In short, EFT is just what we do all the time in physics, some power counting that allow us to estimate the size of different contributions (quite often based simply on dimensional analysis) and a perturbative (Taylor) expansion. This peturbative expansion is of course quite subtle in QFT, due to locality and renormalisation. Contrary to most perturbative expansions in physics, at least in some cases in particle physics we can prove that the corresponding observables are analytic in the relevant variables and, therefore, a Taylor expansion is mathematically consistent. In general, like in any perturbative expansion, how useful it is depends on many factors, including the size of the perturbative parameter or the nature of the expansion itself.

1.2 Observables in particle physics

The main observables in particle physics can be obtained from S-matrix elements which, thanks to the reduction formula [1], can be computed in terms of connected, amputated correlators

$$ {}_{\text{out}}\langle q_1,\dots,q_m|p_1,\dots,p_n\rangle_{\text{in}} = \left(\prod_{i=1}^{m}\sqrt{\mathcal{R}_i}\right)\left(\prod_{j=1}^{n}\sqrt{\mathcal{R}_j}\right) G(q_1,\dots,q_m;p_1,\dots,p_n)\Big|_{\text{conn.,amp.}}, \quad (1) $$

where the correlators are defined by

$$ G(q_1,\dots,q_m;p_1,\dots,p_n) = \prod_{i=1}^{m}\int \mathrm{d}^4y_i \mathrm{e}^{\mathrm{i}q_i\cdot y_i}\prod_{j=1}^{n}\int \mathrm{d}^4x_j \mathrm{e}^{\mathrm{i}p_j\cdot x_j}\langle\Omega|T\{\phi(y_1)\dots\phi(y_m)\phi(x_1)\dots\phi(x_n)\}|\Omega\rangle, \quad (2) $$

and we have considered a single scalar field theory for simplicity. The above equations are valid for any interpolating field

$$ {}_{\text{in}}\langle k|\phi(x)|\Omega\rangle = \sqrt{\mathcal{R}}\mathrm{e}^{\mathrm{i}k\cdot x} \Leftrightarrow D_F(p) = \frac{\mathrm{i}\mathcal{R}}{p^2-m^2}+\dots, \quad (3) $$

where $D_F(p)$ is the full propagator and the dots stand for regular terms in the vicinity of the pole. The independence on the interpolating field will be important when talking about EFT bases below.

Correlators can be computed in perturbation theory, in terms of correlators in free field theory

$$\langle\Omega|T\{\phi(x_1)\dots\phi(x_n)\}|\Omega\rangle = \langle 0|T\{\phi_0(x_1)\dots\phi_0(x_n)\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\,\mathcal{L}_{\rm int}[\phi_0(x)]}\}|0\rangle_{\text{no vac. bubbles}}. \tag{4}$$

These correlators can be computed using Wick's theorem

$$T\{\phi_0(x_1)\dots\phi_0(x_n)\} =: \phi_0(x_1)\dots\phi_0(x_n) : + \text{ all possible contractions}, \tag{5}$$

where every contraction corresponds to a Feynman propagator.

2 Building the effective Lagrangian

In the following we will assume a local, Lorentz invariant, quantum field theory. Furthermore, we will also assume that our theory is weakly coupled, so that a perturbative expansion in loops and operator mass dimension is well behaved. We will also use natural units $\hbar = c = 1$.

We will define our theory by its Lagrangian, the sum of Lorentz (and gauge if relevant) invariant local operators,

$$\mathcal{L} = \sum_i C_i \mathcal{O}_i, \tag{6}$$

where the coefficient of each operator is called Wilson coefficient (WC). Local operators are those operators build with a finite number of fields (all evaluated at the same point) with a finite number of derivatives acting on them. In momentum space the corresponding Feynman rules are polynomials in momenta. Which operators should we use? In principle all local invariant operators should be included. This means an infinite number of operators, which is clearly unmanageable. As we will see in the next section, one can establish power-counting rules that determine the size of the contribution of different operators. Given the finite experimental precision, only operators with a sizeable contribution to experimental observables need to be kept. Furthermore, depending on which physics we want to reproduce, the number of operators to consider can be further reduced.

2.1 Power counting

Power-counting arguments allow us to estimate the size of the contribution of any operator to physical observables.

In perturbation theory, quadratic terms are especially important, as they determine the free-field theory that we expand around in our perturbative calculations. In particular, the kinetic term sets the size of all the remaining scales, via canonical normalization, and the mass dimension of all fields and couplings. The mass term, in turn, fixes the on-shell condition for physical particles. Let's consider the quadratic term of a complex scalar and a Dirac fermion

$$\mathcal{L} = -\phi^\dagger(\partial^2 + m_\phi^2)\phi + \overline{\psi}[\mathrm{i}\partial\!\!\!/ - m_\psi]\psi. \tag{7}$$

Since the action is dimensionless in natural units, the Lagrangian has mass dimension 4. Derivatives (and masses) have mass dimension 1 and therefore we have

$$[\phi] = (4-2)/2 = 1, \tag{8}$$

$$[\psi] = (4-1)/2 = 3/2. \tag{9}$$

This extends to other fields so the mass dimension of any boson is 1 and the one of any fermion is 3/2. This automatically fixes the mass dimension of any local operator, and, therefore, of its WC. Given a local operator of mass dimension D, its WC will have mass dimension $4-D$ and therefore we can generically write our effective Lagrangian

$$\mathcal{L} = \sum_D \mathcal{L}^{(D)}, \tag{10}$$

where $\mathcal{L}^{(D)}$ is the effective Lagrangian of operators of mass dimensions D, generically written as

$$\mathcal{L}^{(D)} = \sum_i C_{D,i} \mathcal{O}_i^{(D)} = \sum_i \frac{c_{D,i}}{\Lambda^{D-4}} \mathcal{O}_i^{(D)}. \tag{11}$$

In the second equality we have introduced an explicit power of the cut-off Λ to make the WC $c_i^{(D)}$ dimensionless. Note that the cut-off represents a physical threshold, like the mass of a heavy particle, and therefore it makes sense to consider a common cut-off for all operators. The corresponding dimensionless WC are expected to be of order one (times the relevant loop suppression) or smaller. Once can devise more sophisticated power-counting rules, depending on the class of UV effects we want our EFT to describe at low energies (see for instance [2] for the case a strongly interacting light Higgs). However, we will stick in these lectures to a purely mass dimension and loop expansion power counting. Thus, our general EFT Lagrangian is given by a double expansion on mass dimension and loop order

$$\mathcal{L}_{\text{EFT}} = \sum_{D\geq 1} \sum_{i=1}^{n_D} \frac{c_{D,i}}{\Lambda^{D-4}} \mathcal{O}_i^{(D)}, \tag{12}$$

with

$$c_{D,i} = \sum_{l\geq 0} \frac{c_{D,i}^{(l)}}{(16\pi^2)^l}, \tag{13}$$

and $c_{D,i}^{(l)}$ expected to be order one, unless generated from a higher scale than Λ or at a higher loop order, in which case it will be smaller than this naive expectation.

Note that for each mass dimension D, the number of invariant operators, assuming a finite number of fields in our theory, is finite. The key observation that makes EFTs useful is that operators of higher mass dimension have smaller contribution to low energy amplitudes than lower-dimensional ones. This applies to the sum of the mass dimension of all operator insertions in the amplitude so adding more higher-dimensional operators is also suppressed. Let's see it in some detail. Let us consider a low scattering amplitude normalised to be dimensionless (the mass dimension of an n-particle scattering amplitude is $4-n$, we can compensate any additional dimension with the available dimensionful scales). We assume that all physical quantities in our theory (physical masses and external momenta, globally denoted by m and p, respectively) are of similar size and much smaller than the cut-off, $p \sim m \ll \Lambda$. Let's assume that the amplitude we are computing has one insertion of an operator of dimension D and all other operators are dimension 4 (so that their coefficients are dimensionless). By dimensional analysis the amplitude has to scale like

$$\mathcal{M} \sim \left(\frac{p}{\Lambda}\right)^{D-4}, \tag{14}$$

where p represents a combination of low energy scales. Thus, we see that, at low energies, the effect of operators of mass dimension larger than 4 are suppresses by the small ratio $(p/\Lambda)^{D-4} \ll$

1. Furthermore, the larger the dimension, the stronger the suppression. The general power counting rule is that

$$\mathcal{M} \sim \left(\frac{p}{\Lambda}\right)^n, \tag{15}$$

where

$$n = \sum_i (D_i - 4), \tag{16}$$

and the sum runs over all operator insertions (vertices or, equivalently WCs) in the diagram. In general, operators of mass dimension larger than 4 have suppressed effects at low energies. [1] Thus, operators of mass dimension larger than a certain value give contribution to experimental observables smaller than the experimental precision and can therefore be excluded from our analysis. Note that this leaves a finite number of operators that we need to include in our EFT. The order in operator dimension (and also in the loop expansion) is fixed by the experimental accuracy for the particular observables we are interested in. For observables with enough experimental precision, the number of operators to include, while finite, can be very large. Our argument seems to be tree-level based but if we use a mass-independent renormalisation scheme, like $\overline{\text{MS}}$ in dimreg, it applies unchanged for loop contributions.

In fact, this argument allows us to discuss the (irr)relevance of operators at low energies according to their mass dimension. Operators of mass dimension larger than 4 are called **irrelevant**, as their contribution at low energies is more and more suppressed; operators of mass-dimension smaller than 4 are called **relevant**, as their importance grows at low energies; finally, operators of mass-dimension 4 are called **marginal**, as their contribution is the same as all energies. As we will see, this scaling gets modified by quantum corrections, via the renormalisation group. In weakly coupled theories, this modification is not crucial for relevant or irrelevant operators but it is for marginal ones, as even a small correction can tilt them in one or the other direction.

2.2 Dimensional analysis in d space-time dimensions

As already mentioned, we will use dimreg to regularise our loop integrals. It is therefore convenient to extend our previous power-counting arguments, based on dimensional analysis, to the case of arbitrary $d = 4 - 2\varepsilon$ dimensions. The procedure is similar to the one in the previous section but now the Lagrangian has mass dimension d in d dimensions. Derivatives (and masses) still have mass dimension 1 in any number of space-time dimensions and therefore we have

$$[\phi] = (d-2)/2 = [\phi]_{d=4} - \varepsilon, \tag{17}$$
$$[\psi] = (d-1)/2 = [\psi]_{d=4} - \varepsilon, \tag{18}$$

where $[\phi]_{d=4} = 1$ and $[\psi]_{d=4} = 3/2$. The same happens for other fields so that, independently of whether they are bosons or fermions, we have

$$[\Phi]_d = [\Phi]_{d=4} - \varepsilon, \tag{19}$$

[1] Note that, in principle, the contribution of an operator of mass-dimension larger than 4 could be compensated by operators of mass-dimension smaller than 4. Adding more of these smaller than 4 mass dimension operators would then seem to compensate arbitrary higher-dimensional operators. In practice, even if we expect these operators to be proportional to a positive power of the cut-off, Λ, they are observed to be of the order of the light scales -which is related to the so-called hierarchy problem- and therefore they cannot compensate the contribution of higher-dimensional operators.

where Φ stands for any field. This automatically allows us to obtain the mass dimension of any operator (and therefore its WC). Let us consider a generic term in the Lagrangian of the form

$$\mathcal{L} = C\,\mathcal{O} = C\,\partial^m \Phi^n, \tag{20}$$

where we have symbolically represented an operator with n fields (of any kind) and m derivatives acting on them. Requiring that the mass dimension of the Lagrangian is d we get

$$4 - 2\varepsilon = [C] + [\mathcal{O}]_{d=4} - n\varepsilon \Rightarrow [C] = 4 - [\mathcal{O}]_{d=4} + (n-2)\varepsilon = [C]_{d=4} + (n-2)\varepsilon. \tag{21}$$

We can preserve dimensional analysis as in $d = 4$ by including explicitly a dimensionful scale in the WCs,

$$\mathcal{L} = \mu^{(n-2)\varepsilon} C\,\mathcal{O}, \tag{22}$$

where n is the number of fields (of any kind) appearing in $\mathcal{O}$ and now $[C]_d = [C]_{d=4}$.

Note:-

This is consistent with redefining the integral measure when going to $d = 4 - 2\varepsilon$ dimensions to maintain the mass dimension of the corresponding integral for any d

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \to \mu^{4-d} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \equiv \mu^{2\varepsilon} \int_k, \tag{23}$$

where we have defined a notation that we will heavily use in this notes for the $d-$dimensional loop integral. The radial measure is then

$$\mu^{2\varepsilon} k^{d-1} \mathrm{d}k = \left(\frac{\mu}{k}\right)^{2\varepsilon} k^3 \mathrm{d}k \ll k^3 \mathrm{d}k, \quad \text{for } \mu \ll k \ (\varepsilon > 0). \tag{24}$$

Thus, for positive ε the integrals are effectively cut-off in the UV at scales

$$\Lambda_{\text{dimreg}} \sim \mu^2 \mathrm{e}^{\frac{1}{\varepsilon}} \gg \mu. \tag{25}$$

2.3 Regularisation and renormalisation

In QFT we have to integrate over all values of undetermined loop momenta. This gives rise to divergent integrals that can be dealt with via regularisation and renormalisation. In these lectures we will use dimensional regularisation (dimreg) to regulate the integrals and modified minimal substraction ($\overline{\text{MS}}$) to renormalise them. Mass independent regularisation schemes, like dimreg, have the advantage to preserve the power counting of the EFT in terms of mass dimensions, as we will discuss below. We will assume that the reader is comfortable with dimreg and $\overline{\text{MS}}$ and discuss here only the properties that are relevant for us.

Some useful properties of dimreg (for one-loop matching calculations) are

- Scaleless integrals vanish. By scaleless integrals we mean integrals that only depend on the loop momentum, up to a (possibly dimensionful) proportionality factor. Of these, a special one is

$$\int_k \frac{1}{k^4} = \frac{\mathrm{i}}{16\pi^2}\left(\frac{1}{\varepsilon_{\text{UV}}} - \frac{1}{\varepsilon_{\text{IR}}}\right) = 0, \tag{26}$$

  by which we mean that this particular integral vanishes because of the cancellation (after analytic continuation) between a UV and an IR pole, which is relevant for renormalisation (in which the only relevant contribution is the UV pole). All other scaleless integrals vanish identically (not involving UV or IR poles).

Exercise 2.1. Argue by dimensional analysis that all scaleless integrals except possibly the one given in Eq.(26) are identically zero. Using the identity

$$\frac{1}{k^4} = \frac{1}{k^2(k^2-M^2)} - \frac{M^2}{k^4(k^2-M^2)} \tag{27}$$

split the integral in Eq. (26) in two separate integrals, with the integrands in the previous identity, and discuss their UV and IR divergences. Compute them using dimreg and prove the identity in (26). You can use the general result

$$I_{n,m} \equiv \int_k \frac{1}{(k^2)^n}\frac{1}{(k^2-M^2)^m} = \frac{(-1)^{n+m}\mathrm{i}}{(4\pi)^{2-\varepsilon}(M^2)^{n+m-2+\varepsilon}}\frac{\Gamma(n+m-2+\varepsilon)\Gamma(2-n-\varepsilon)}{\Gamma(m)\Gamma(2-\varepsilon)}. \tag{28}$$

Solution:-

A generic scaleless integral of the form

$$I_n \equiv \mu^{-2\varepsilon}\int_k 1/k^n, \tag{29}$$

has mass dimension $4-n$. If $n \neq 4$ then it has to be proportional to a dimensionful scale, but there is no dimensionful scale in the integral so it has to be identically 0 by dimensional analysis. The case $n = 4$ is special but we can write

$$I_4 = \mu^{-2\varepsilon}\int_k \frac{1}{k^4} = \mu^{-2\varepsilon}\int_k \frac{1}{k^2(k^2-M^2)} - \mu^{-2\varepsilon}\int_k \frac{M^2}{k^4(k^2-M^2)} \equiv I_4^{\mathrm{UV}} - I_4^{\mathrm{IR}}. \tag{30}$$

where the first integral is UV divergent but IR convergent and the opposite for the second one. Using Eq. (28) we obtain

$$I_4^{\mathrm{UV}} = I_{1,1} = \frac{\mathrm{i}}{16\pi^2}\left(\frac{1}{\bar{\varepsilon}_{\mathrm{UV}}} + 1 + \ln\frac{\mu^2}{M^2}\right), \tag{31}$$

$$I_4^{\mathrm{IR}} = M^2 I_{2,1} = \frac{\mathrm{i}}{16\pi^2}\left(\frac{1}{\bar{\varepsilon}_{\mathrm{IR}}} + 1 + \ln\frac{\mu^2}{M^2}\right). \tag{32}$$

Technically we have $\varepsilon_{\mathrm{UV}} > 0$ and $\varepsilon_{\mathrm{IR}} < 0$ but we can analytically continue one into the other and have a unique ε. Taking the difference we obtain the requested result.

- Integration by parts identities

$$\int_k \frac{1}{(k^2-M^2)^{n+1}} = \frac{d-2n}{2n}\frac{1}{M^2}\int_k \frac{1}{(k^2-M^2)^n}, \quad n \geq 1, \tag{33}$$

which allow us to write all relevant integrals in terms of the tadpole integral, with value

$$\mu^{2\varepsilon}\int_k \frac{1}{(k^2-M^2)} = \frac{\mathrm{i}M^2}{16\pi^2}\left[\frac{1}{\bar{\varepsilon}} + 1 + \log\left(\frac{\mu^2}{M^2}\right) + \mathcal{O}(\varepsilon)\right], \tag{34}$$

where $1/\bar{\varepsilon} \equiv 1/\varepsilon - \gamma_E + \log(4\pi)$, with $\gamma_E \approx 0.577$ the Euler-Mascheroni constant.

Exercise 2.2. Use the identity

$$0 = \int_k \frac{\partial}{\partial k^\mu} \frac{k^\mu}{(k^2 - M^2)^n}, \tag{35}$$

to prove Eq. (33).

Solution:-

$$\begin{aligned} \frac{\partial}{\partial k^\mu} \frac{k^\mu}{(k^2 - M^2)^n} &= \frac{d}{(k^2 - M^2)^n} - \frac{2nk^2}{(k^2 - M^2)^{n+1}} \\ &= \frac{d}{(k^2 - M^2)^n} - \frac{2n(k^2 - M^2 + M^2)}{(k^2 - M^2)^{n+1}} \\ &= \frac{d - 2n}{(k^2 - M^2)^n} - \frac{2nM^2}{(k^2 - M^2)^{n+1}}, \end{aligned} \tag{36}$$

which, upon integration produces Eq. (33).

Some other properties are not particular of dimreg but will be very useful in our calculations. They include

- Partial fractioning

$$\frac{1}{(k^2 - m_1^2)(k^2 - m_2^2)} = \frac{1}{m_1^2 - m_2^2} \left[\frac{1}{k^2 - m_1^2} - \frac{1}{k^2 - m_2^2} \right], \quad m_1 \neq m_2, \tag{37}$$

  which allow us to separate propagators with different masses.

- Hard region expansion. Different exact identities allow us, upon iteration, to perform expansions in the so-called hard region limit, in which the loop momentum is much larger than the external momenta or the masses of the external (light) particles.

$$\frac{1}{(k+p)^2 - M^2} = \frac{1}{k^2 - M^2} \left[1 - \frac{p^2 + 2k \cdot p}{(k+p)^2 - M^2} \right], \tag{38}$$

  which is useful to expand in the limit $p^2 \ll k^2 \sim M^2$, the hard region in the presence of heavy masses.

$$\frac{1}{(k+p)^2 - m^2} = \frac{1}{k^2} \left[1 - \frac{p^2 + 2k \cdot p - m^2}{(k+p)^2 - m^2} \right], \tag{39}$$

  which corresponds to the hard region in the presence of light masses ($p^2 \sim m^2 \ll k^2$). Note that in this case the degree of divergence in the UV is preserved but the one on the IR is worsened. Thus, in general spurious IR divergences can be generated in this process. These spurious IR divergences are related to spurious UV divergences in other kinematic limits of the same integral which can be related to UV divergences in the EFT. At one loop IR poles can be easily disentangled from UV ones and there is usually no problem whenever we are interested in the renormalisation of our theory. If we go beyond one loop or we need to find the finite rational terms that come from UV poles it is generally more difficult to disentangle IR from UV poles and another identity is normally useful [3]

$$\frac{1}{(k+p)^2 - m^2} = \frac{1}{k^2 - M_\Lambda^2} \left[1 - \frac{2k \cdot p + p^2 + M_\Lambda^2 - m^2}{(k+p)^2 - m^2} \right], \tag{40}$$

where M_Λ is a spurious mass that regulates all IR divergences. Upon iteration we will obtain eventually a convergent integral that does not contribute any $1/\varepsilon$ poles. Any remaining poles are necessarily UV and we only need to keep those that produce a result that is independent of the spurious mass.

2.4 Locality of UV divergences and renormalisation

One loop amplitudes can have IR ($k \to 0$) or UV ($k \to \infty$) divergences. Both are regularised in dimreg but they have very different origins and consequences. Roughly speaking, IR divergences are non-physical, and they cancel in sufficiently exclusive physical observables. UV divergences, instead, are cancelled when we express physical observables in terms of other physical observables. This is very cumbersome in practice and it is much easier to use the WCs and the fields in our Lagrangian as intermediaries. This is possible because UV divergences, once subdivergences have been dealt with, are local (a polynomial in momenta) and can therefore be absorbed in the WC of local operators. Locality of UV divergences can be simply proven by taking derivatives of the corresponding integral with respect to external momenta. Each integral will improve the UV divergence of the integral so that after a finite number of derivatives we end up with a finite integral. The original one can be recovered by integration so that the divergences are located in the integration constants, that contribute as a polynomial in momenta and are therefore local.

Exercise 2.3. Consider the following divergent integral

$$I(p) = \int_0^\infty \mathrm{d}k \frac{k}{k+p}. \tag{41}$$

Take as many derivatives with respect to p as needed to make the integral finite and compute it. Write the original integral, by integrating the result with respect to p, as a non-local function of p plus a polynomial in p (with possibly divergent coefficients).

Solution:-

$I(p)$ is linearly divergent in the UV ($k \to \infty$). Its first integral is logarithmically divergent and the second one is finite,

$$I'(p) = -\int_0^\infty \mathrm{d}k \, \frac{k}{(k+p)^2}, \quad I''(p) = 2\int_0^\infty \mathrm{d}k \, \frac{k}{(k+p)^3} = \frac{1}{p}. \tag{42}$$

Integrating twice, with respect to p, we obtain

$$\int\int \mathrm{d}p I''(p) = \int \mathrm{d}p \, [c_1 + \ln p] = p \ln p - p + c_1 p + c_2. \tag{43}$$

where c_1 and c_2 are integration constants that contain the UV divergences of the original integral. The important point is that they arise from the integration and therefore are always propostional to polynomial in momenta, a sign of local operators generating them.

The practical idea behind renormalisation is that, since UV divergences are local (after non-local subdivergences have been subtracted), they can be eliminated in a redefinition of the WCs appearing in the Lagrangian (technically fields also need to be redefined to absorb

divergences proportional to the kinetic term). Since these objects (WCs and fields) are not directly observable, this redefinition can be simply viewed as fixing them by experimental measurements of physical observables. Once they have been fixed by experiment, we can compute any physical observable in terms of them and the result is guaranteed to be finite.

In order to simplify our notation we are going to neglect two loop and higher orders in perturbation theory. Subtleties arising at these orders with respect to the one-loop case we are going to consider, like the presence of non-local divergences due to UV divergent subdiagrams, can be dealt with by systematically renormalising the theory to lower orders in the loop expansion. It is best to work with the quantum effective action, $\Gamma[\Phi]$, the generator of one-particle-irreducible (1PI) Green's functions. Since the vertices arising from Γ are to be used in tree-level diagrams, no further loop divergences appear once Γ has been made finite via renormalisation.

Once subdivergences have been subtracted (which is automatically done at a certain loop orded by renormalising the theory at lower loops), the remaining UV divergences are local so that we can write the divergent part of the quantum action in terms of a divergent local Lagrangian,

$$\Gamma_{div} = \int \mathrm{d}^d x \, \mathcal{L}_{\text{div}} = \int \mathrm{d}^d x \sum_i C_i' \mathcal{O}_i, \tag{44}$$

where C_i' are divergent coefficients parametrising the UV divergences (plus maybe some local terms, depending on the chosen renormalisation scheme). The parameters in the Lagrangian defining our model (WCs and fields) are not directly observable but can be fixed in terms of observables that depend on them. This process of fixing them involves the cancellation of possible divergences and fixing the finite part to reproduce experimental measurements. The former can be systematised by means of a renormalisation scheme. In these lectures we will use the $\overline{\text{MS}}$ scheme, in which only poles in $1/\bar{\varepsilon}$ are subtracted. The simplest way to do this is to split the original (bare) objects into renormalised ones (finite, to be fixed by experiment) and counterterms (infinite, with possibly a finite part that depends on the renormalisation scheme, that cancel the divergences in the quantum action) as follows. First, we will canonically normalise the divergent Lagrangian appearing in Eq. (44) so that the kinetic term is normalised to one. This eliminates the need for a wave-function renormalisation counterterm and our fields can then be already considered the fields in the canonically normalised theory are already renormalised fields. We then split the WCs into a renormalised coupling and a counterterm, that will eliminate the relevant divergences [2]

$$C_i^{\text{bare}} = \mu^{(n_i-2)\varepsilon}(C_i + \delta C_i). \tag{47}$$

[2] Our notation is completely equivalent to a more standard notation in which we do not canonically normalise $\mathcal{L}_{\text{div}}$ and introduce the following renormalisation constants

$$C_i^{\text{bare}} = \mu^{(n_i-2)\varepsilon} Z_i C_i, \quad \Phi^{\text{bare}}(x) = \sqrt{Z_\Phi}\Phi(x), \tag{45}$$

where Z_i are the WC renormalisation constants and Z_Φ the wave function renormalisation constant (for simplicity we have not distinguished between the different fields appearing in the operators but there should be a factor of $\sqrt{Z_\Phi}$ for each field in $\mathcal{O}_i$). The bare Lagrangian can then be written again in terms of a renormalised Lagrangian plus counterterms by expanding the renormalisation constants

$$\mathcal{L} = \sum_i C_i^{\text{bare}} \mathcal{O}_i^{\text{bare}} = \sum_i \mu^{(n_i-2)\varepsilon} Z_i Z_\Phi^{\frac{n_i}{2}} C_i \mathcal{O}_i = \mu^{(n_i-2)\varepsilon} C_i \mathcal{O}_i + \text{counterterms}, \tag{46}$$

and renormalisation proceeds like in our case.

Assuming that we have canonically normalised the divergent Lagrangian in Eq.(44) we then have

$$\delta C_i = -C_i'. \tag{48}$$

2.5 Renormalisation group equations

The process of renormalisation introduces a renormalisation scale, μ, that appears in the calculation of loop amplitudes. Of course, this is an arbitrary scale that we have introduced and physical observables can therefore not depend on it. This means that renormalised WCs should also depend on μ in such a way that physical observables are independent of μ. The simplest way of computing the dependence of renormalised WCs on μ is to use the fact that bare WCs do not depend on μ so that we get the scale dependence from the counterterms [3]. Let's see how this is done in general. Let's define the beta function of a Wilson coefficient

$$\beta_{C_i} \equiv \dot{C}_i \equiv \frac{\mathrm{d}C_i}{\mathrm{d}\ln\mu}. \tag{49}$$

We then have

$$\begin{aligned}\frac{\mathrm{d}C_i^{\text{bare}}}{\mathrm{d}\ln\mu} = 0 &= \mu^{(n_i-2)\varepsilon}\left[\varepsilon(n_i-2)(C_i+\delta C_i) + \beta_i + \frac{\partial \delta C_i}{\partial C_j}\beta_j\right] \\ &= \mu^{(n_i-2)\varepsilon}[\varepsilon(\overrightarrow{nC} + \overrightarrow{n\delta C}) + (\mathbb{1}+M)\cdot\vec{\beta}],\end{aligned} \tag{50}$$

where we have used matrix notation in the last equality with the following definitions

$$\overrightarrow{nC}_i \equiv (n_i-2)C_i, \tag{51}$$

$$\overrightarrow{n\delta C}_i \equiv (n_i-2)\delta C_i, \tag{52}$$

$$M_{ij} \equiv \frac{\partial \delta C_i}{\partial C_j}, \tag{53}$$

$$\vec{\beta}_i \equiv \beta_i. \tag{54}$$

From Eq.(50) we can write a formal expression for the beta functions, valid to any loop order,

$$\vec{\beta} = -\varepsilon(\mathbb{1}+M)^{-1}\cdot(\overrightarrow{nC} + \overrightarrow{n\delta C}). \tag{55}$$

Note that in perturbation theory the inverse of the corresponding matrix is well defined in a perturbative sense.

Let us now consider the ε expansion of the above equation. The counterterms (and therefore the matrix M) have an expansion

$$\delta C_i = \sum_{n=0}^{\infty}\frac{C_i^{(n)}}{\varepsilon^n}, \quad M = \sum_{n=0}^{\infty}\frac{M^{(n)}}{\varepsilon^n}. \tag{56}$$

It is therefore clear that the beta functions have a generic expansion of the form

$$\beta_i = \sum_{n=-1}^{\infty}\frac{\beta_i^{(n)}}{\varepsilon^n} = \beta_i^{(-1)}\varepsilon + \beta_i^{(0)} + \dots\ . \tag{57}$$

[3] We are grateful to Renato Fonseca for numerous discussions on this issue, that have shaped the procedure described below.

The beta function in four dimensions is $\beta_i^{(0)}$ and more singular terms are normally considered to vanish, giving consistency conditions of the calculation at 2 and higher loops (see however [4]). Introducing the expansions, Eq.(56) into the general expression for the beta function and keeping the finite term in the $1/\varepsilon$ expansion we obtain the correct result. In order to get a closed expression let us consider the beta functions in the MS scheme (we can go from this to the $\overline{\text{MS}}$ scheme by simply taking $\mu^2 \to \tilde{\mu}^2 = 4\pi \mathrm{e}^{-\gamma_E}\mu^2$. In that case we have $C_i^{(0)} = M^{(0)} = 0$ and therefore

$$\vec{\beta} = -\varepsilon \left[\overrightarrow{nC} + \frac{\overrightarrow{n\delta C} - M^{(1)} \cdot \overrightarrow{nC}}{\varepsilon} + \dots \right] = -\varepsilon\, \overrightarrow{nC} + M^{(1)} \cdot \overrightarrow{nC} - \overrightarrow{n\delta C} + \mathcal{O}(1/\varepsilon), \tag{58}$$

so that $\beta_i^{(-1)} = -(n_i - 2)C_i$ and,

$$\beta_i^{(0)} = (n_j - 2)C_j \frac{\partial \delta C_i^{(1)}}{\partial C_j} - (n_i - 2)\delta C_i^{(1)}. \tag{59}$$

Finally, using the relation between the counterterms and the WCs of the divergent Lagrangian, Eq. (48), we obtain the final expression for the beta functions

$$\beta_i^{(0)} = \left[(n_i - 2) - (n_j - 2)C_j \frac{\partial}{\partial C_j} \right] C_i'^{(1)} = -2LC_i'^{(1)}, \tag{60}$$

where we have defined

$$C_i' = \sum_{n=1}^{\infty} \frac{C_i'^{(n)}}{\varepsilon^n}, \tag{61}$$

and also the loop operator L that, when acting on a perturbative expression gives, for each term in the expanded expression, the loop order of the term times the term itself.

Exercise 2.4. Use the following topological identities for connected diagrams

$$L = P - V + 1, \quad 2P + E = \sum_v n_v, \tag{62}$$

where the sum in the second expression runs over all vertices in the diagram, L stands for the number of loops, P for the number of internal propagators, V for the number of vertices, n_v is the number of particles on vertex v and E is the number of external particles, to show that

$$\left[(n_i - 2) - (n_j - 2)C_j \frac{\partial}{\partial C_j} \right] C_{i,D}'^{(1)} = -2LC_{i,D}'^{(1)}. \tag{63}$$

Where $C_{i,D}'^{(1)}$ is a polynomial of WCs arising from diagram D.

Solution:-

$C_i'^{(1)}$ is computed as a sum of contributions from Feynman diagrams

$$C_i'^{(1)} = \sum_D C_{i,D}'^{(1)}, \tag{64}$$

where D runs over all diagrams that contribute and $C_{,D}i'^{(1)}$ is a polynomial in all the

WCs of the model. Using the topological identities we have

$$\left[(n_i-2)-(n_j-2)C_j\frac{\partial}{\partial C_j}\right]C'^{(1)}_{i,D}=\left[E-2-\sum_v(n_v-2)\right]C'^{(1)}_{i,D}$$
$$=\left[E-2-2P-E+2V\right]C'^{(1)}_{i,D}=-2LC'^{(1)}_{i,D}. \quad (65)$$

In the first identity we have used that $n_i = E$ and that

$$C_j\frac{\partial}{\partial C_j}, \quad (66)$$

acting on $C'^{(1)}_{i,D}$ just runs over all vertices returning $C'^{(1)}_{i,D}$.

2.6 Dimensional analysis and renormalisability

The power-counting argument used in the previous sections can be used to sketch the relation between the mass dimension of the operators in the Lagrangian and renormalisability. Going back to the general power-counting rule in Eqs. (15) and (16) we see that the insertion of operators of mass dimension less than four reduce the dimension of the corresponding divergence (they contribute positive powers of Λ). Thus, they need counterterms of lower mass dimension. These operators are called super-renormalisable. Insertion of operators of mass-dimension 4 maintain the mass dimension fo the divergence. They are called renormalisable. Insertion of operators of mass-dimension larger than 4, however, increase the degree of the divergence (insertion of two dimension 5 operators induce a divergence of dimension 6) and therefore need counterterms of higher dimension than the operators themselves. These are called non-renormalisable operators. The situation is clear. Renormalisable (and super-renormalisable) theories can only generate divergences of mass-dimension 4 or lower, which can therefore be rernormalised with a finite number of counterterms. Non-renormalisable theories, on the other hand, generate divergences of higher and higher dimensions by introducing more insertions of non-renormalisable operators and therefore need an infinite number of counterterms to renormalise the theory. [4]

The important message we want to pass on is that, while non-renormalisable need, in principle, an infinite number of counterterms, in practice we only care about the effective Lagrangian up to a finite mass dimension. Counterterms of higher dimension can therefore be discarded and, for all practical calculations, non-renormalisable theories can be renormalised in exactly the same way remormalisable theories are.

Note that there is a direct correlation between super-renormalisable, renormalisable and non-renormalisable operators with relevant, marginal and irrelevant operators. There was a time in which only renormalisable (including super-renormalisable interactions) theories were considered valid. This is not the case anymore and we now tend to consider everything an EFT. It is still true, however, because of this relation, that at energies much lower than the physical

[4] The situation is actually a bit more involved than the simple sketch we have described here, as the insertion of more vertices typically lead to a larger number of propagators and, therefore, to a more convergent integral (unless there are powers of momenta in the vertices themselves). The conclusion is nevertheless true. A more detailed discussion that considers the superficial degree of divergence can be found in any QFT textbook.

threshold, only (super-)renormalisable interactions are relevant for phenomenology. One needs to approach the phsycial threshold or increase the precision of the experimental measurements, to be sensitive to the effects of non-renormalisable interactions.

3 EFT bases

The basis of operators that we need to use in our EFT depends on the physics we want to describe. As we mentioned above, the experimental precision of physical observables determine the maximum operator dimension we need to keep in our basis. Operators that are related to others by integration by parts relations are redundant, and can be eliminated from our basis (integration by parts corresponds physically to momentum conservation). Similarly, group theory identities can be used to further reduce the number of operators needed. No further reduction can be done in general, and the resulting basis, called Green's basis, can be used to compute off-shell quantities in d dimensions. Gauge invariance can be tricky here, because it is broken by gauge fixing when quantizing the theory (only BRST invariance is preserved, which is enough to ensure gauge invariance of physical observables but not of un-physical off-shell quantities, like general off-shell Green's functions or counterterms). The solution is to either add non-gauge-invariant operators to our Green's basis or, much more conveniently, use the background field method, that we will discuss below. If we are doing calculations that are essentially in $d = 4$ dimensions, like tree-level calculations or one-loop RGEs (that are done in $d = 4 - 2\varepsilon$ dimensions but we are only interested in the coefficient of the $1/\varepsilon$ UV pole and therefore the rest of the amplitude can be computed in 4 dimensions), we can further simplify our basis by using 4-dimensional properties, like Fierz identities. Finally, if we are computing on-shell physical observables we can further reduce our Green's basis to a physical basis, in which some operators that are needed for off-shell calculations become redundant in on-shell observables. This class of redundant operators can be reduced by using field redefinitions, or equivalently, at the linear level, equations of motion (EOM).

Building a physical basis, a Green's basis and the reduction of the latter into the former is a non-trivial task but it is a taks that, in principle, only needs to be done once. One is free to use different (physical or Green's) bases, and several have been proposed in the case of the Standard Model EFT (SMEFT). Some are more convenient for comparing with experimental data, some for connecting them with models of new physics. If complete, they are all equivalent, provided all the relevant translations between bases are done correctly.

3.1 On-shell redundancies: field redefinitions and equations of motion

We have emphasized that S-matrix elements can be computed, via the reduction formula, from on-shell correlators of arbitrary interpolating fields. It is then clear that we can perform perturbative field redefinitions, that maintain the nature of the interpolating fields, to reduce certain operators in the Green's basis in favour of the ones in the physical one. In the path integral formalism of QFT the fields are integration variables in the path integral and field redefinitions just correspond to change of variables. The actual reduction of operators goes beyond a simple change of variables, as it corresponds to a change in the Lagrangian without the corresponding change in the source terms or the Jacobian. A careful analysis shows that this can always be done to reduce redundant operators [5, 6].

We will see a very simple example in the following exercise

Exercise 3.1. Consider the following Lagrangian for a real scalar

$$\mathcal{L} = \mathcal{L}_4 + \mathcal{L}_6, \tag{67}$$

$$\mathcal{L}_4 = -\frac{1}{2}\phi(\partial^2 + m^2)\phi - \lambda\phi^4, \tag{68}$$

$$\mathcal{L}_6 = \frac{\alpha_{61}}{\Lambda^2}\phi^6 + \frac{\beta_{62}}{\Lambda^2}\phi^3\partial^2\phi. \tag{69}$$

Perform the field redefinition $\phi \to \phi + \beta_{62}\phi^3/\Lambda^2$ and obtain the Lagrangian in terms of the new interpolating field. Compute now the EoM for ϕ from $\mathcal{L}_4$ and introduce the solution on the second operator of $\mathcal{L}_6$. Compare the resulting Lagrangians. Compute now the effect of the field redefinition up to dimension 8. Compute the EoM for ϕ using the full $\mathcal{L}$ and insert the solution back in the second operator of $\mathcal{L}_6$. Compare the resulting Lagrangians.

Solution:-

After the field redefinition the Lagrangian goes to

$$\begin{aligned}\mathcal{L} \to & -\frac{1}{2}\phi(\partial^2 + m^2)\phi - \left(\lambda + \beta_{62}\frac{m^2}{\Lambda^2}\right)\phi^4 \\ & + \frac{1}{\Lambda^2}\left(\alpha_{61} - 4\lambda\beta_{62} - \frac{\beta_{62}^2}{2}\frac{m^2}{\Lambda^2}\right)\phi^6 + \frac{1}{\Lambda^4}\left[(6\alpha_{61}\beta_{62} - 6\lambda\beta_{62}^2)\phi^8 + \frac{39}{10}\beta_{62}^2\phi^5\partial^2\phi\right],\end{aligned} \tag{70}$$

where we have used integration by parts to prove the following two identities

$$\phi^4(\partial_\mu\phi)(\partial^\mu\phi) = -\frac{1}{5}\phi^5\partial^2\phi, \tag{71}$$

$$\phi^3\partial^2\phi^3 = \frac{9}{5}\phi^5\partial^2\phi. \tag{72}$$

The EoM from $\mathcal{L}$ are given by

$$\partial^2\phi = -m^2\phi - 4\lambda\phi^3 + 6\frac{\alpha_{61}}{\Lambda^2}\phi^5 + \frac{\beta_{62}}{\Lambda^2}(3\phi^2\partial^2\phi + \partial^2\phi^3). \tag{73}$$

When inserted in the redundant operator, we obtain

$$\begin{aligned}\frac{\beta_{62}}{\Lambda^2}\phi^3\partial^2\phi \to & \frac{\beta_{62}}{\Lambda^2}\phi^3\left[-m^2\phi - 4\lambda\phi^3 + 6\frac{\alpha_{61}}{\Lambda^2}\phi^5 + \frac{\beta_{62}}{\Lambda^2}(3\phi^2\partial^2\phi + \partial^2\phi^3)\right] \\ = & -\beta_{62}\frac{m^2}{\Lambda^2}\phi^4 - 4\lambda\beta_{62}\frac{1}{\Lambda^2}\phi^6 + 6\alpha_{61}\beta_{62}\frac{1}{\Lambda^4}\phi^8 + \frac{24}{5}\beta_{62}^2\frac{1}{\Lambda^4}\phi^5\partial^2\phi.\end{aligned} \tag{74}$$

Comparing Eq. (74) with Eq (70) we see that the use of EoM correctly recovers the effect of the field redefinitions at leading order, written in blue in Eq. (74), while the terms quadratic in β_{61} come with the incorrect coefficient or are even not present.

A few comments are in order:

- As mentioned, field redefinitions are allowed provided we are computing physical observables. In this case we can use field redefinitions to eliminate redundant operators from

the Green's basis to obtain the physical basis.

- Note that, in the construction of the physical basis we can always use EoM, order by order in the $1/\Lambda$ expansion. We might miss contributions that are generated at higher orders but we do not care about those because we will start at the next order with the full Green's basis (with arbitrary WCs). If we care about the detailed value of the WCs, for instance if we are matching a full model onto our EFT, as we will describe below, then we are forced to use field redefinitions (unless quadratic terms are higher order than we need to consider).

- This result (redundant operators can be reduced via field redefinitions) is not a tree-level statement but holds to all orders in perturbation theory.

- One can understand the equivalence between the two theories diagrammatically, in which applying the EoM (or using field redefinitions) can be seen as dressing the external legs with further interactions. We will give more details below, when we discuss on-shell matching.

3.2 Redundancies in $d = 4$: evanescent structures

New redundancies, that allow us to further reduce the operators in our basis, appear in $d = 4$ dimensions. Examples are Fierz identities or the fact that in 4 dimensions there is a finite basis of fermion bilinears (similarly for relations with contracted levi-civita tensors). These properties are however not fulfilled in general $d = 4 - 2\varepsilon$ dimensions, generating what are usually called evanescent operators. These are operators that are formally $\mathcal{O}(\varepsilon)$ and therefore are irrelevant for tree-level matching and one-loop RGE calculations but they can hit a UV pole (only UV poles contribute because they correspond to a local effect) and result in a finite, rational term with physical effect starting at one loop for finite matching calculations and two loops for RGEs. There are several ways of dealing with evanescent operators. Similarly, there is an infinite number of ways of defining the evanescent operators themselves (we can add an arbitrary $\mathcal{O}(\varepsilon)$ contribution to them remaining evanescent). These different ways fix the evanescent scheme. For the moment, we will focus on the simpler effects at one-loop order and describe a simple way to include them in the finite one-loop matching. We do it with an example taken from [7].

Consider the following two operators

$$(\mathcal{O}_{le})_{prst} = (\bar{l}_p\gamma^\mu l_r)(\bar{e}_s\gamma_\mu e_t), \quad (\mathcal{R}_{le})_{prst} = (\bar{l}_p e_r)(\bar{e}_s l_t), \tag{75}$$

where $prst$ are flavour indices, l is the SM lepton doublet and e is the SM charged-lepton singlet. In 4 dimensions they are related by a Fierz identity but this is no longer true in general d dimensions so we can define an evanescent operator as follows

$$(\mathcal{R}_{le})_{prst} = -\frac{1}{2}(\mathcal{O}_{le})_{ptsr} + (\mathcal{E}_{le})_{prst}, \tag{76}$$

where the evanescent operator $\mathcal{E}_{le}$ vanishes in 4 dimensions and the above equation defines it in arbitrary d dimensions.

We need to care about these evanescent structures whenever one of the two operators are generated but the result is given in terms of the other. This can be because they belong to two different physical bases and one needs to translate the results of the matching from one basis

to the other or because one is generated in d dimensions but it is not in the physical basis in 4 dimensions, so the results of the matching are given in terms of the other. Let's see an example of the latter case. Indeed, the operator $\mathcal{O}_{le}$ is in the most standard basis for the SMEFT at dimension 6, the Warsaw basis [8], while $\mathcal{R}_{le}$ is not. Following [7] we consider an extension of the SM with a heavy copy of the Higgs boson that, for simplicity, only couples to leptons

$$\mathcal{L} = \mathcal{L}_{SM} + D_\mu \Phi^\dagger D^\mu \Phi + M^2 \Phi^\dagger \Phi - (y^{pr}_{\Phi e} \bar{l}_p \Phi e_r + \text{h.c.}). \tag{77}$$

Integrating out (we will see below how to do it) Φ at tree level we get

$$(c_{le})_{prst} = \frac{y^{pr}_{\Phi e} (y^{ts}_{\Phi e})^*}{M^2}, \tag{78}$$

where c_{le} denotes the WC of $\mathcal{R}_{le}$ and we will denote by C_{le} the one of $\mathcal{O}_{le}$. Going to the physical basis we would then say that

$$(C_{le})_{prst} = -\frac{1}{2}(c_{le})_{ptsr}. \tag{79}$$

However, while this is correct at the tree-level, it loses the information that in d dimensions it was in fact $\mathcal{R}_{le}$ the operator that was generated, and this has effects at one-loop order. Indeed, if we were to insert the operator $\mathcal{R}_{le}$ in one-loop diagrams, it would not generate dipole operators for the leptons, as it does not have enough number of gamma matrices. $\mathcal{O}_{le}$ on the other hand, has enough gamma matrices and indeed it generates a (spurious in our case) dipole operator at one loop. The evanescent contribution precisely cancels this spurious contribution so that the calculations in the EFT reproduce the effect of having generated $\mathcal{R}_{le}$ at tree level. In order to systematically compute the effect we just need to insert the evanescent operator $\mathcal{E}_{le}$ in the EFT in all possible one-loop amplitudes, and compute the finite term that arises from the product of a UV $1/\varepsilon$ pole times the $\mathcal{O}(\varepsilon)$ contribution of the evanescent operator. Doing so for the amplitude corresponding to the dipole operator with the hypercharge gauge bosons is

$$\Delta(C_{eB})_{pr} = \frac{3 g_Y y^{ts}_e}{128\pi^2} (c_{le})_{prst}. \tag{80}$$

Thus, we can give the result of the matching in two different (but equivalente ways),

$$\left.\begin{aligned} &(c_{le})_{prst} = \tfrac{y^{pr}_{\Phi e}(y^{ts}_{\Phi e})^*}{M^2}, \\ &(C_{le})_{prst} = 0, \\ &\Delta(C_{eB})_{pr} = 0, \end{aligned}\right\} \text{ basis in } d \text{ dimensions}, \tag{81}$$

$$\left.\begin{aligned} &(C_{le})_{prst} = -\tfrac{1}{2}\tfrac{y^{pt}_{\Phi e}(y^{rs}_{\Phi e})^*}{M^2}, \\ &\Delta(C_{eB})_{pr} = \tfrac{3 g_Y y^{ts}_e}{128\pi^2}(c_{le})_{prst}. \end{aligned}\right\} \text{ basis in 4 dimensions}, \tag{82}$$

where we have only included the extra evanescent contribution to the electron-hypercharge dipole operator when the results are given in the 4-dimensional basis (note that this is, in principle, independent of whether the matching is performed on-shell or off-shell). There can be also direct one-loop contributions to this (and other) dipole operator and also other evanescent contributions. All the relevant evanescent contributions in the SMEFT at dimension 6 and one-loop order have been computed in [7].

3.3 Gauge invariance: the background field method

Non-abelian gauge theories introduce an extra complication when computing local contributions from UV physics (let it be renormalisation or matching the effects of heavy physics at low energies). The main point is that in order to quantise gauge theories we need to fix the gauge, after which our theory is no longer gauge invariant, but just BRST invariant. This is enough to guarantee gauge invariant results for physical observables but not for intermediate, non-physical objects like counterterms of off-shell Green's functions. In order to solve this problem, and simplify our intermediate calculations, we can resort to the background field method. We will introduce the main ideas here but full details can be found, for instance in [9].

The generating functional of a non-abelian gauge hteory reads

$$Z[J] = \int \mathcal{D}Q_\mu \, \det\left[\frac{\delta G^a}{\delta \omega^b}\right] \, \exp \mathrm{i}\left[S[Q] - \frac{1}{2\xi} G\cdot G + J\cdot Q\right], \tag{83}$$

where the dot denotes an integral over space-time, G^a is the gauge fixing term (for instance we can use $G^a = \partial_\mu Q^{a\,\mu}$), and the determinant is of the derivative of the gauge fixing action with respect to the gauge parameter of a gauge transformation

$$\delta Q^a_\mu = -f^{abc}\omega^b Q^c_\mu + \frac{1}{g}\partial_\mu \omega^a = \frac{1}{g}(D_\mu \omega)^a. \tag{84}$$

Let us now define a background field generating functional

$$\widetilde{Z}[J,A] = \int \mathcal{D}Q_\mu \, \det\left[\frac{\delta \widetilde{G}^a}{\delta \omega^b}\right] \, \exp \mathrm{i}\left[S[Q+A] - \frac{1}{2\xi} \widetilde{G}\cdot \widetilde{G} + J\cdot Q\right], \tag{85}$$

where the infinitesimal gauge transformation is now

$$\delta Q^a_\mu = -f^{abc}\omega^b (Q^c_\mu + A^c_\mu) + \frac{1}{g}\partial_\mu \omega^a = \frac{1}{g}(D_\mu \omega)^a, \tag{86}$$

with $\widetilde{G} = \widetilde{G}[Q,.A]$ in general. Making the change of variables $Q \to Q - A$ we can see that

$$\begin{aligned}\widetilde{Z}[J,A] &= \int \mathcal{D}Q_\mu \, \det\left[\frac{\delta \widetilde{G}'^a}{\delta \omega^b}\right] \, \exp \mathrm{i}\left[S[Q] - \frac{1}{2\xi} \widetilde{G}'\cdot \widetilde{G}' + J\cdot (Q-A)\right] \\ &= \mathrm{e}^{-\mathrm{i}J\cdot A}\widetilde{Z}'[J,0] = \mathrm{e}^{-\mathrm{i}J\cdot A} Z'[J],\end{aligned} \tag{87}$$

where

$$\widetilde{G}'[Q,A] = \widetilde{G}[Q-A,A], \tag{88}$$

and the prime denotes this new gauge fixing function.

We can now define the associated generator of connected Green's functions

$$\widetilde{W}[J,A] = -\mathrm{i}\ln \widetilde{Z}[J,A] = -J\cdot A + W'[J], \tag{89}$$

with

$$\widetilde{Q} = \frac{\delta \widetilde{W}}{\delta J} = -A + \frac{\delta W'}{\delta J} = -A + \overline{Q}'. \tag{90}$$

Finally, we can define the corresponding 1PI background field quantum effective action

$$\widetilde{\Gamma}[\widetilde{Q},A] = \widetilde{W}[J,A] - J\cdot \widetilde{Q} = W'[J] - J\cdot \overline{Q}' = \Gamma'[\overline{Q}] = \Gamma'[\widetilde{Q}+A]. \tag{91}$$

In particular

$$\Gamma'[A] = \widetilde{\Gamma}[0, A]. \tag{92}$$

Thus, in order to compute the 1PI quantum action, we just need to compute $\widetilde{\Gamma}[0, A]$. Several comments are in order:

- The gauge choice for $\widetilde{\Gamma}$ and Γ' are different, but this does not matter for physical observables.
- When computing $\widetilde{\Gamma}[0, A]$ we only integrate over Q and therefore these quantum fields are the only ones that can appear in loops. Furthermore, we set $\widetilde{Q}$ to zero so the quantum fields Q cannot appear as external particles. The bottom line is that we use $S[Q + A]$ to compute the Feynman rules and then compute 1PI amplitudes with A as external states and Q running only in loops. If we were to compute physical amplitudes, A can also run in bridges (tree level progpagators) stitching 1PI vertices together.
- As we have emphasized, the issue of gauge-fixing is relevant for off-shell, unphysical objects. If we perform an on-shell matching, using directly physical, on-shell amplitudes, we do not need to use the background field method to obtain a gauge invariant result.

The main point of this procedure is that, since we don't need propagators of the background A fields in order to compute the quantum action, we don't need to fix the gauge for them. Thus, we can simply fix the gauge for the quantum fields and leave the background gauge symmetry unbroken. For instance, if we choose

$$\widetilde{G}^a = \partial_\mu Q^{a\,\mu} + g f^{abc} A^b_\mu Q^{c\,\mu} \equiv \hat{D}_\mu Q^{a\,\mu}, \tag{93}$$

where the hat denotes covariant derivative with respect to the background gauge field. It can then be seen that $\widetilde{\Gamma}[Q, A]$ is invariant under a background gauge transformation

$$\delta A^a_\mu = \frac{1}{g} \hat{D}_\mu \omega^a, \tag{94}$$

with all other fields (Q^a_μ, c^a, $\overline{c}^a$) transforming in the adjoing under the background gauge transformation (and possible matter fields transforming under the corresponding representation). All this is straight-forward for ungroken gauge symmetries. Spontaneously broken gauge symmetries are trickier, in particular when matching calculations are involved. See Ref. [10] for the latest on the subject.

What happens if we insist on now using the background field method (or equivalently, if we use a gauge fixing that is not background field gauge invariant)? What happens is that the quantum effective action will in general receive non-gauge invariant contributions. These contributions will be redundant on-shell but have to be maintained in any intermediate off-shell calculation. As an example, if we use the background field method but choose a gauge-fixing term of the form

$$\mathcal{L}_{\rm GF} = (\partial_\mu G^{a\,\mu})^2, \tag{95}$$

that indeed breaks the background gauge symmetry, the one-loop quantum action reads

$$\Gamma = -\frac{1}{4}(\hat{F}^a_{\mu\nu})^2 + \frac{g^2}{16\pi^2} C_A \frac{1}{\varepsilon} \left[\frac{10}{3} (\hat{F}^a_{\mu\nu})^2 - 4 g f^{abc} \hat{F}^a_{\mu\nu} \hat{A}^{b\,\mu} \hat{A}^{c\,\nu} \right]. \tag{96}$$

This contribution is indeed not gauge invariant but the extra term can be written in the following way

$$-gf^{abc}\hat{F}^a_{\mu\nu}\hat{A}^{b\,\mu}\hat{A}^{c\,\nu} = (\hat{F}^a_{\mu\nu})^2 + 2\hat{A}^{a\,\nu}\hat{D}^\mu\hat{F}^a_{\mu\nu} - 2\partial_\mu(\hat{A}^a_\nu F^{a\,\mu\nu}) \overset{\text{on-shell}}{\longrightarrow} (\hat{F}^a_{\mu\nu})^2, \tag{97}$$

where in the on-shell limit we have used taht the second term vanishes and the third one is a surface term that does not contribute in perturbation theory.

3.4 Bases for the SMEFT

Obtaining a basis (physical or Green's, in 4 or d dimensions) is a non-trivial task. It took many years to get a complete physical basis in 4 dimensions for the SMEFT at mass-dimension 6, since the initial attempt [11] to the basis that the community has taken as the standard one, the Warsaw basis [8]. Going from that basis to the corresponding Green's one took another decade [12] and it was not until very recently that all the relevant evanescent structures and contributions, for the reduction of a complete basis in d dimensions were computed in [7]. The development of new methods and the automation via computer codes, like `BasisGen` [13] or `sym2int` [14, 15] has allowed for a much quicker development of higher-dimensional bases, including [16, 17, 18] at dimension 7, [19, 20] at dimension 8, and [21] at dimension 9.

4 Using EFTs: bottom-up vs top-down

One of the (many) great advantages of EFTs is that they allow to obtain the low energy effects of heavy physics in a very efficient, two-step, process. In a first step, commonly called the bottom-up approach, we compute experimental observables in terms of the WCs of the EFT. Note that this is a fairly model-independent process. Indeed, the EFT is able to capture the low-energy physical effects of any model that has, as the only light fields and relevant symmetries at low energies, the ones of the EFT and arbitrary heavy physics, with a significant mass gap between the heavy particles and the energies at which we are performing our experimental measurements. Thus, we can see this bottom-up approach as a very convenient, model-independent, parametrisation of experimental data.

The second step, called the top-down approach, we compute the WCs of the EFT for specific full models with heavy particles, in terms of the couplings and masses of the full theory. This calculation is called matching, as the WCs can be computed by matching certain amplitudes in the full theory and the EFT. It is sometimes also called integrating out the heavy physics, as this is what one does explicitly when using the path integral formalism.

4.1 Matching full theories onto EFTs: general ideas

There are several different ways of matching a full theory onto the corresponding EFT. In this section we are going to focus on the diagrammatic approach, which can itself be performed in two different ways, off-shell and on-shell.

Off-shell matching, the most commonly performed in the literature, is done by computing 1lPI Green's functions, with only light particles as external legs, both in the full theory and the EFT, at the required loop order. Expanding both in ratios of light scales over heavy masses, and considering arbitrary off-shell kinematics, the difference is local and can therefore be parametrised by the WCs of local operators in our EFT. It is enough to compute 1lPI Green's

functions because this way we are able to reconstruct the light quantum effective action, which is an intrinsically off-shell object and therefore requires the off-shell matching. Some of the advantages of this off-shell matching is that the required number of diagrams is relatively small and that the off-shell kinematics allows for a large degreee of redundancy that provides non-trivial cross-checks of the results. Fuerthermore, as we will see in a moment, the expansion by regions method allows us to obtain directly the matching coefficients by computing some limit (the hard region contribution) of the full theory, without having to do the corresponding calculation in the EFT. The disadvantages are that a Green's basis is needed and that the background field method has to be implemented when gauge theories are present.

We can alternatively perform the matching directly on-shell, so that no redundant operators need to be considered and the background field method is not required. This is done by matching all amputated, connected physical (on-shell) amplitudes with light external particles. The difference between the corresponding amplitudes in the (expanded) full theory and the EFT is again local and can be parametrised this time in terms of the WCs of a physical basis. The advantage of not requiring redundant operators is usually countered by the fact that a much larger number of diagrams (now including 1-particle-reducible ones, with light bridges) havs to be included and the fact that, even the hard region expansion of the full theory is non-local, and the non-localities cancel between the calculation in the full theory and the EFT in a non-trivial way.

4.2 Expansion by regions and locality of the matching conditions

The general idea of EFTs is to perform a Taylor expansion in the (small) ratio of light scales over heavy ones. In particular, denoting generically with p arbitrary linear combinations of the external momenta of our light particles and by m their masses, while M stands for the generic mass of a heavy particle ($p \sim m \ll M$) and k will stand for (combinations of) loop momenta. The tree level propagator of a heavy particle can then be generically expanded as follows

$$\frac{1}{p^2 - M^2} = -\frac{1}{M^2} \sum_{n=0}^{\infty} \left(\frac{p^2}{M^2} \right)^n . \tag{98}$$

Now, if the propagator is part of a one-loop diagram, then it involves loop momenta, that run fron 0 to infinity. Which expansion can we do in that case? The answer is that, in dimensional regularisation, we can split a momentum integral in the sum of several integrals, each dominated by a so-called region, in which the integral is divergent, and expanded in a power series in the corresponding limit. This expansion by regions extends to any loop order but we will see it explicitly in a very simple example at one-loop order, to simplify the discussion.

Consider the following integral

$$\begin{aligned} I_{\rm F} &= \mu^{2\varepsilon} \int_k \frac{1}{k^2 - M^2} \frac{1}{k^2 - m^2} = \frac{\mathrm{i}}{16\pi^2} \left\{ \frac{1}{\bar{\varepsilon}} + 1 + \ln \frac{\mu^2}{M^2} + \frac{m^2}{M^2 - m^2} \ln \frac{m^2}{M^2} \right\} \\ &= \frac{\mathrm{i}}{16\pi^2} \left\{ \frac{1}{\bar{\varepsilon}} + 1 + \ln \frac{\mu^2}{M^2} + \left(\frac{m^2}{M^2} + \frac{m^4}{M^4} + \dots \right) \ln \frac{m^2}{M^2} \right\}, \end{aligned} \tag{99}$$

where in the last line we have expanded, after performing the integral, in the small ration $m^2/M^2 \ll 1$. The integrand is divergent in two regions,

$$m \sim p \sim k \ll M, \quad \text{soft region}, \tag{100}$$

$$m \sim p \ll k \sim M, \quad \text{hard region}, \tag{101}$$

where in this particular example we do not have external momenta but they can be incorporated without problems. The expansion by regions method tells us that we can compute the integral by considering separately both regions, expanding the integrand in the small ratios, integrating over the full integration range and adding both integrals. Let us see that indeed this is the case in our example. Before we use partial fractioning to write our integral in a slightly simpler form for the different expansions

$$I_F = \frac{\mu^{2\varepsilon}}{M^2-m^2}\int_k \left(\frac{1}{k^2-M^2}-\frac{1}{k^2-m^2}\right). \tag{102}$$

Exercise 4.1. Starting from Eq. (102) compute the integral to reproduce the result in Eq. (99).

Solution:-

$$\begin{aligned} I_F &= \frac{\mu^{2\varepsilon}}{M^2-m^2}\int_k \left(\frac{1}{k^2-M^2}-\frac{1}{k^2-m^2}\right) \\ &= \frac{1}{M^2-m^2}\frac{\mathrm{i}}{16\pi^2}\left(M^2\left[\frac{1}{\bar{\varepsilon}}+1-\ln\frac{M^2}{\mu^2}\right]-m^2\left[\frac{1}{\bar{\varepsilon}}+1-\ln\frac{m^2}{\mu^2}\right]\right) \\ &= \frac{\mathrm{i}}{16\pi^2}\left(\frac{1}{\bar{\varepsilon}}+1+\ln\mu^2-\frac{M^2}{M^2-m^2}\ln M^2+\frac{m^2}{M^2-m^2}\left[\ln m^2+\ln M^2-\ln M^2\right]\right) \\ &= \frac{\mathrm{i}}{16\pi^2}\left(\frac{1}{\bar{\varepsilon}}+1+\ln\frac{\mu^2}{M^2}+\frac{m^2}{M^2-m^2}\ln\frac{m^2}{M^2}\right). \end{aligned} \tag{103}$$

Where we have denoted in blue a term that has been added and subtracted to get the final result.

We start with the soft region expansion, in which $k\sim m\sim p\ll M$.

$$\begin{aligned} I_F^{\text{soft}} &= \frac{\mu^{2\varepsilon}}{M^2-m^2}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\left[\left(-\frac{1}{M^2}\right)\left(1+\frac{k^2}{M^2}+\frac{k^4}{M^4}+\dots\right)-\frac{1}{k^2-m^2}\right] \\ &= \frac{-\mathrm{i}}{16\pi^2}\frac{m^2}{M^2-m^2}\left(\frac{1}{\bar{\varepsilon}}+1+\ln\frac{\mu^2}{m^2}\right). \end{aligned} \tag{104}$$

Note that this soft region contribution is in general non-local in the light scales (in this example in the light mass m but in general external momenta will also appear inside logarithms and other non-polynomial functions). We now consider the hard region, in which $m\sim p\ll k\sim M$

$$\begin{aligned} I_F^{\text{hard}} &= \frac{\mu^{2\varepsilon}}{M^2-m^2}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\left[\frac{1}{k^2-M^2}-\left(\frac{1}{k^2}\right)\left(1+\frac{m^2}{k^2}+\frac{m^4}{k^4}+\dots\right)\right] \\ &= \frac{\mathrm{i}}{16\pi^2}\frac{M^2}{M^2-m^2}\left(\frac{1}{\bar{\varepsilon}}+1+\ln\frac{\mu^2}{M^2}\right). \end{aligned} \tag{105}$$

Note that we now get in general a non-local dependence on the heavy mass M, but the light scales, including the light masses and external momenta, as they have been expanded out, only result in a local contribution. In both calculations we have made heavy use of the fact that scaleless integrals vanish in dimreg.

Adding both integrals we obtain

$$I_F^{\rm soft} + I_F^{\rm hard} = \frac{\rm i}{16\pi^2}\left\{\frac{1}{\bar{\varepsilon}} + 1 + \frac{1}{M^2-m^2}\left(m^2\ln\frac{m^2}{\mu^2} + M^2\ln\frac{\mu^2}{M^2}\right)\right\}$$
$$= \frac{\rm i}{16\pi^2}\left\{\frac{1}{\bar{\varepsilon}} + 1 + \ln\frac{\mu^2}{M^2} + \frac{m^2}{M^2-m^2}\ln\frac{m}{M^2}\right\} = I_F. \tag{106}$$

This result is completely general, the full integral is the sum of the integrals expanded in all the relevant regions. At one loop we only have two regions, the soft region (that includes in general the deep IR) and the hard region (that includes the deep UV). At higher loops the different regions can be more complicated but it is still true that the full integral can be obtained as the sum of the corresponding regions. The apparent double counting that we might incur in due to the integration over the full range is not present thanks again to the fact that scaleless integrals vanish in dimreg.

We are now in a position to sketch the proof that the difference between the full theory amplitudes and the EFT ones are local and therefore can be matched to the local EFT Lagrangian. Indeed, the soft region corresponds to expanding the integrand in ratios of all scales, including the loop momenta, over the heavy masses -before the loop integration-. This expansion precisely reproduces the tree-level expansion of the EFT and therefore the soft region contribution of the one-loop amplitude in the full theory exactly coincides with the full one-loop amplitude in the EFT. The difference between the full one-loop amplitude in the full theory and the one in the EFT thus corresponds to the hard region contribution of the one-loop amplitude in the full theory. As we have argued, this one is always local (as we have expanded in external momenta up to a finite order and it is therefore a polynomial in external momenta, the contribution of a local operator). This proof extends to higher loop orders. The arguments is more involved because of the mixed hard/soft contributions at two and higher loop orders. These mixed contributions correspond to EFT loops to which the lower order WCs contribute, and are taking into account by matching at lower loop order. This is in a similar spirit to the non-local subdivergences that are cancelled by lower order counterterms in the renormalisation process. The net result is that the contribution to the l-loop order matching corresponds, once the lower order contributions in the EFT from the matching at lower loop orders have been taken into account, to the all hard region contribution and it is therefore also local. A more detailed discussion can be found in [22].

Some comments about the expansion by regions are in order and they are already exemplified in the above example.

- We have seen that the soft region contribution in the full theory corresponds to the full contribution in the EFT. [5] This is consistent with the expansion by regions in the EFT itself since, as there are no heavy scales in the EFT, the hard region expansion always corresponds to scaleless contributions that vanish. Thus, we have in general

$$I_{\rm F} - I_{\rm EFT} = I_{\rm F}^{\rm soft} + I_{\rm F}^{\rm hard} - I_{\rm EFT}^{\rm soft} - I_{\rm EFT}^{\rm hard} = I_{\rm F}^{\rm soft} + I_{\rm F}^{\rm hard} - I_{\rm EFT}^{\rm soft} = I_{\rm F}^{\rm hard}. \tag{107}$$

- The expansion by regions introduces in general spurious $1/\varepsilon$ divergences (that cancel among the different regions). These can be relevant if we are interested in the divergent

[5] Technically, this is strictly true if we do the matching in d dimensions. If we match in $d = 4$ dimensions then there might be differences between the soft region contribution in the full theory and the EFT calculation. These differences correspond to an evanescent effect and can in fact be used to compute the relevant evanescent shifts, as we will describe below in the introduction to on-shell matching.

parts of our amplitudes. In general, the soft region expansion contains the correct IR poles in our integral but generate, in general, spurious UV poles. The hard region, instead, correctly reproduces the UV poles but generate spurious IR poles.

- The UV poles of the EFT are related (in fact identical) to the IR poles in the matching calculation. Let's see it in detail. In general the amplitude in the EFT can be written (before renormalisation) as,

$$I^{\mathrm{E}} = \frac{A^E}{\varepsilon_{UV}} + \frac{B}{\varepsilon_{IR}} + C^E = \left[\frac{A^E}{\varepsilon_{UV}} + \frac{B}{\varepsilon_{IR}} + C^E\right] + \left[\frac{A^E}{\varepsilon_{UV}} - \frac{A^E}{\varepsilon_{IR}}\right], \tag{108}$$

  where the first term is square brackets corresponds to the soft contribution and the second to the hard one (in this case the latter is scaleless and therefor vanishes). In the full theory, instead, we have

$$I^{\mathrm{F}} = \frac{A^F}{\varepsilon_{UV}} + \frac{B}{\varepsilon_{IR}} + C^F = \left[\frac{D^F}{\varepsilon_{UV}} + \frac{B}{\varepsilon_{IR}} + C_s^F\right] + \left[\frac{A^F}{\varepsilon_{UV}} - \frac{D^F}{\varepsilon_{IR}} + C_h^F\right]. \tag{109}$$

  Using now that the soft regions agree in both amplitudes we obtain

$$A^E = D^F, \quad C^E = C_s^F. \tag{110}$$

  The first equality indeed tells us that the UV pole in the EFT (A^E) is equal to minus the IR pole in the hard region contribution of the full theory D^F and therefore of the matching calculation. This can be used to re-sum IR logarithms in the full theory, by turning them into UV logs in the EFT and use the EFT RGE to re-sum them.

- The UV poles in both theories are in general different and they have to be renormalised independently.

- We have kept the UV divergences explicit in our previous discussion to understand the divergence structure of the full and EFT calculations but, when computing the matching, we should first renormalise (independently) both theories and then subtract only the renormalised amplitudes. These might still be IR divergent but, as we have seen, we the IR divergences are identical in both theories and cancel in the matching. Thus, we see that we can get the matching contributions by computing the hard region contribution to the amplitude in the full theory, neglecting the $1/\bar{\varepsilon}$ terms, either because they are UV, and thus renormalised away, or IR and thus cancelled in the difference.

- Doing the matching by computing the hard region contribution of the amplitudes in the full theory has the extra advantage that the calculation itself is much easier than computing the full amplitude. Indeed, at one loop order partial fractioning and ibp identities allow us to write all amplitudes in terms of a single integral, the tadpole one defined in Eq. (34). Going to higher loops we need more integrals but still the number of the required master integrals is much smaller than if we were to compute the full amplitudes.

- Another effect of this simplified matching procedure is that this method automatically intertwines the EFT calculation with the one in the full theory. In principle one can do the matching by computing the complete full theory amplitude, expand in the heavy scales (after integrating) and subtract the full EFT calculation. In that case both calculations can be done independently, including different renormalisation schemes, if desired.

5 Off-shell matching (diagrammatic)

How do we perform diagrammatic off-shell matching in a systematic way? We have to use the following algorithm, that will be explained and justified in the following sections:

1. Build a Green's basis. This has to be done only once, and, if you are lucky, someone has probably done it for you.

2. Compute, at tree level, all the off-shell amplitudes that are needed to fix the WCs of the EFT (again, this can be done just once per EFT). Since we need to compute only 1lPI contributions, these correspond simply to the local contributions from the vertices in the EFT.

3. Compute the hard region (in which all loop momenta and the masses of the heavy particles are much heavier than the external momenta and light masses) of the one-light-particle-irreducible (1lPI) contribution to all the relevant amplitudes in the full theory.

4. Equate these amplitudes to the corresponding ones in the EFT computed at tree level, match independently all different kinematic configurations (imposing momentum conservation) to fix the values of the WCs.

5. Non-trivial checks of the calculation come from the off-shell consistency for different kinematic configurations and also from gauge invariance (that relates amplitudes with a different number of particles (in which gauge bosons are replaced with external momenta).

In the following sections we will describe in detail this procedure using a very simple example. Let us consider the following full theory

$$\mathcal{L}_{\text{full}} = \bar{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi - \frac{1}{2}\phi(\partial^2 + m^2)\phi - \eta\bar{\psi}\psi\phi - \frac{1}{2}\Phi(\partial^2 + M^2)\Phi - \lambda\bar{\psi}\psi\Phi, \tag{111}$$

where we assume $m \ll M$ and, for simplicity, we have taken the mass of the light fermion and scalar equal (but this is irrelevant for the discussions below). The idea is to systematically find, up to one-loop order, the EFT that reproduces the effects of this model at energies much smaller than M. We have used the word systematic several times already, and indeed the procedure is systematic, but in our examples we will only do part of the complete matching and running, focusing on specific groups of operators.

The first step is to build a Green basis, that we will use to match off-shell. The reduction to the physical basis can be performed after the matching has been done. In the following, we will discuss the results, leaving most of the detailed calculations as exercises. Let us focus on the four-fermion operators, up to mass dimension 8. A suitable Green's basis can be chosen as follows

$$\mathcal{L}_{\text{EFT}} = \frac{c_{\psi^4}}{2\Lambda^2}\mathcal{O}_{\psi^4} + \frac{1}{\Lambda^4}\left\{\left[c^{(1)}_{d^2\psi^4}\mathcal{O}^{(1)}_{d^2\psi^4} + \text{h.c.}\right] + \sum_{i=2}^{3} c^{(i)}_{d^2\psi^4}\mathcal{O}^{(i)}_{d^2\psi^4}\right\}, \tag{112}$$

where the different operators read

$$\mathcal{O}_{\psi^4} = (\bar{\psi}\psi)(\bar{\psi}\psi) \qquad \begin{aligned} \mathcal{O}^{(1)}_{d^2\psi^4} &= (\bar{\psi}\psi)(\bar{\psi}\partial^2\psi), \\ \mathcal{O}^{(2)}_{d^2\psi^4} &= (\bar{\psi}\psi)(\partial_\mu\bar{\psi}\partial^\mu\psi), \\ \mathcal{O}^{(3)}_{d^2\psi^4} &= (\bar{\psi}\partial_\mu\psi)(\partial^\mu\bar{\psi}\psi). \end{aligned} \tag{113}$$

Figure 1: 1lPI contribution at tree level to the $\psi\psi \to \psi\psi$ amplitude in the full theory.

Recall that, when finding the Green's basis, we have used integration by parts to reduce operators. Note that this is not a complete Green's basis, not even at dimensions 6, as we can always have terms with gamma matrices. It is enough, however, for the scalar theory that we are considering.

5.1 Tree level matching

In order to match our full theory onto the EFT we can simply compute an off-shell $\psi\psi \to \psi\psi$ amplitude. Remember that we only have to consider connected, 1lPI contributions to the amplitude. The corresponding diagrams are given in Fig. 1. And their contribution reads

$$\begin{aligned} \mathrm{i}\mathcal{M}_F &= \bar{u}_3 u_1 \bar{u}_4 u_2 (-\mathrm{i}\lambda)^2 \frac{\mathrm{i}}{(p_3 - p_1)^2 - M^2} - (3 \leftrightarrow 4) \\ &= \bar{u}_3 u_1 \bar{u}_4 u_2 \frac{\mathrm{i}\lambda^2}{M^2} \left(1 + \frac{p_1^2 + p_3^2 - 2p_1 \cdot p_3}{M^2} + \mathcal{O}\left(\frac{p^4}{M^4}\right)\right) - (3 \leftrightarrow 4), \end{aligned} \tag{114}$$

where the $-(3 \leftrightarrow 4)$ stands for the contribution of the crossed diagram and in the second line we have expanded in external momenta over the heavy mass. The result in the EFT reads

$$\begin{aligned} \mathrm{i}\mathcal{M}_E = \mathrm{i}\bar{u}_4 u_1 \bar{u}_3 u_2 \Big\{ & C_{\psi^4} - C^{(1)}_{d^2\psi^4}[p_1^2 + p_2^2] - (C^{(1)}_{d^2\psi^4})^*[p_3^2 + p_4^2] \\ & + C^{(2)}_{d^2\psi^4}[p_1 \cdot p_3 + p_2 \cdot p_4] + C^{(3)}_{d^2\psi^4}[p_1 \cdot p_4 + p_2 \cdot p_3] \Big\} - (3 \leftrightarrow 4), \end{aligned} \tag{115}$$

where we have reabsorbed the corresponding power of the cut-off in the definition of the (now dimensionful) WCs

$$c_{\psi^4} \equiv C_{\psi^4}\Lambda^2, \quad c^{(i)}_{d^2\psi^4} \equiv C^{(i)}_{d^2\psi^4}\Lambda^4. \tag{116}$$

Equating the coefficient of every kinematic invariant we obtain an over-constrained linear system of equations, with the unique solution

$$C_{\psi^4} = \frac{\lambda^2}{M^2}, \quad C^{(1)}_{d^2\psi^4} = -\frac{\lambda^2}{2M^4}, \quad C^{(2)}_{d^2\psi^4} = -\frac{\lambda^2}{M^4}, \quad C^{(3)}_{d^2\psi^4} = 0. \tag{117}$$

Exercise 5.1. Match the two amplitudes and show that indeed the system has a unique solution.

Solution:-

$$
\begin{aligned}
\mathrm{i}\mathcal{M}_E = \mathrm{i}\bar{u}_4 u_1 \bar{u}_3 u_2 \Big\{ & C_{\psi^4} - C^{(1)}_{d^2\psi^4}[p_1^2 + p_2^2] - (C^{(1)}_{d^2\psi^4})^*[p_3^2 + p_4^2] \\
& + C^{(2)}_{d^2\psi^4}[p_1 \cdot p_3 + p_2 \cdot p_4] + C^{(3)}_{d^2\psi^4}[p_1 \cdot p_4 + p_2 \cdot p_3] \Big\} \\
= \bar{u}_4 u_1 \bar{u}_3 u_2 \mathrm{i} \Big\{ & C_{\psi^4} + \big[C^{(3)}_{d^2\psi^4} - C^{(1)}_{d^2\psi^4} - \big(C^{(1)}_{d^2\psi^4}\big)^*\big] p_1^2 + \big[C^{(2)}_{d^2\psi^4} - C^{(1)}_{d^2\psi^4} - (C^{(1)}_{d^2\psi^4})^*\big] p_2^2 \\
& - 2\big(C^{(1)}_{d^2\psi^4}\big)^* p_3^2 + \big[C^{(2)}_{d^2\psi^4} + C^{(3)}_{d^2\psi^4} - 2\big(C^{(1)}_{d^2\psi^4}\big)^*\big] p_1 \cdot p_2 \\
& + \big[C^{(2)}_{d^2\psi^4} - C^{(3)}_{d^2\psi^4} + 2\big(C^{(1)}_{d^2\psi^4}\big)^*\big] p_1 \cdot p_3 \\
& + \big[- C^{(2)}_{d^2\psi^4} + C^{(3)}_{d^2\psi^4} + 2\big(C^{(1)}_{d^2\psi^4}\big)^*\big] p_2 \cdot p_3 \Big\} - (3 \leftrightarrow 4),
\end{aligned} \tag{118}
$$

where in the second equality we have used momentum conservation to eliminate $p_4 = p_1 + p_2 - p_3$. The term that survives in the limit of vanishing momenta gives

$$
C_{\psi^4} = \frac{\lambda^2}{M^2}. \tag{119}
$$

Equating the terms proportional to p_3^2 we get

$$
C^{(1)}_{d^2\psi^2} = -\frac{\lambda^2}{2M^4}. \tag{120}
$$

Equating now the one proportional to p_1^2 we get

$$
C^{(3)}_{d^2\psi^2} = \frac{\lambda^2}{M^4} + C^{(1)}_{d^2\psi^2} + (C^{(1)}_{d^2\psi^2})^* = 0. \tag{121}
$$

Finally, using the term proportional to p_2^2 we get

$$
C^{(2)}_{d^2\psi^2} = +C^{(1)}_{d^2\psi^2} + (C^{(1)}_{d^2\psi^2})^* = -\frac{\lambda^2}{M^4}. \tag{122}
$$

It is easy to see that all other terms are correctly reproduced with these values.

5.2 Tree-level matching (functionally)

As we will discuss elsewhere, matching can be also done functionally in the path integral formalism. At tree level this is particularly simple, as it just corresponds to computing the classical EoM for the heavy fields, replacing the solution back in the Lagrangian and expanding in the heavy scales. Let's see how this works in our simple example. The EoM for our heavy field are

$$
(\partial^2 + M^2)\Phi = -\lambda \bar{\psi}\psi, \tag{123}
$$

with formal solution

$$
\Phi = \Pi(-\lambda \bar{\psi}\psi), \tag{124}
$$

where we have defined the inverse quadratic term

$$\Pi \equiv (\partial^2 + M^2)^{-1} = \frac{1}{M^2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\partial^2}{M^2}\right)^n. \tag{125}$$

Inserting the solution back in the Lagrangian we get,

$$\mathcal{L} \to \frac{\lambda^2}{2}(\bar{\psi}\psi)\Pi(\bar{\psi}\psi) + \ldots = \frac{\lambda^2}{2}\left[\frac{(\bar{\psi}\psi)^2}{M^2} - \frac{(\bar{\psi}\psi)\partial^2(\bar{\psi}\psi)}{M^4} + \ldots\right]. \tag{126}$$

The dots in the first equality corresponds to the original Lagrangian involving only light fields and the ones on the second equality correspond to higher-dimensional operators. Expanding the derivatives we can check that the resulting effective Lagrangia agrees with the one we obtained with our diagrammatic calculation.

5.3 One-loop running

Once we have computed the tree level matching, we are going to compute the one-loop running in the EFT. We do this, instead of going directly to the one-loop matching because of several reasons. The first one is that we can compute the one-loop running of the EFT independently of any full theory that completes it in the UV. Second, when the relevant scales (matching and experiment) are wildly separated, large logarithms can disrupt the fixed-order perturbative expansion and we need to use one-loop RGEs in connection with tree-level matching to complete the leading order (LO) in the RGE-improved perturbative expansion (more on this below). Finally, the one-loop calculation of the RGEs is similar in spirit to the finite one-loop matching but simpler, as we only need to match the divergent pieces and there are no relevant evanescent effects to worry about. In order to simplify the discussion, we will focus on the fermionic EFT up to mass-dimension 6 operators. Thus, the relevant part of our EFT Lagrangian reads

$$\mathcal{L}_{\rm EFT} = \mathcal{L}_{d\leq 4} + \frac{C}{2}(\bar{\psi}\psi)^2, \tag{127}$$

where recall that $[C] = -2$. We want to compute the beta function for C so we need to renormalise the four-fermion operator itself. For that, we need to compute the divergences appearing, again, in $\psi\psi \to \psi\psi$ scattering. Note that, since we want to compute the effect up to dimension 6, we can only have one insertion of C. The relevant diagrams (the crossed ones are not explicitly displayed) are shown in Fig. 2.

Let's compute them. Since we are only interested in the UV divergences at mass dimension 6 we can expand in all the light scales (light mass m and external momenta) which, in practice means neglecting them. Setting to zero exernal momenta and the light masses we obtain,

$$\begin{aligned}
\mathrm{i}\mathcal{M}_a &= \frac{\mathrm{i}}{16\pi^2\varepsilon}(-2\eta^2 C)\ \bar{u}_4 u_2 \bar{u}_3 u_1 + \text{ finite},\\
\mathrm{i}\mathcal{M}_b &= \frac{\mathrm{i}}{16\pi^2\varepsilon}\left(\frac{\eta^2 C}{2}\right)\bar{u}_4\gamma^\mu u_2 \bar{u}_3\gamma_\mu u_1 + \text{ finite},\\
\mathrm{i}\mathcal{M}_c &= \frac{\mathrm{i}}{16\pi^2\varepsilon}\left(-\frac{\eta^2 C}{2}\right)\bar{u}_4\gamma^\mu u_2 \bar{u}_3\gamma_\mu u_1 + \text{ finite}.
\end{aligned} \tag{128}$$

Thus, the total divergence reads

$$\mathrm{i}\mathcal{M}_{\rm div} = \mathrm{i}\mathcal{M}_{a\,\rm div} - (3 \leftrightarrow 4) = \frac{\mathrm{i}}{16\pi^2\varepsilon}(-2\eta^2 C)\ \bar{u}_4 u_2 \bar{u}_3 u_1 - (3 \leftrightarrow 4). \tag{129}$$

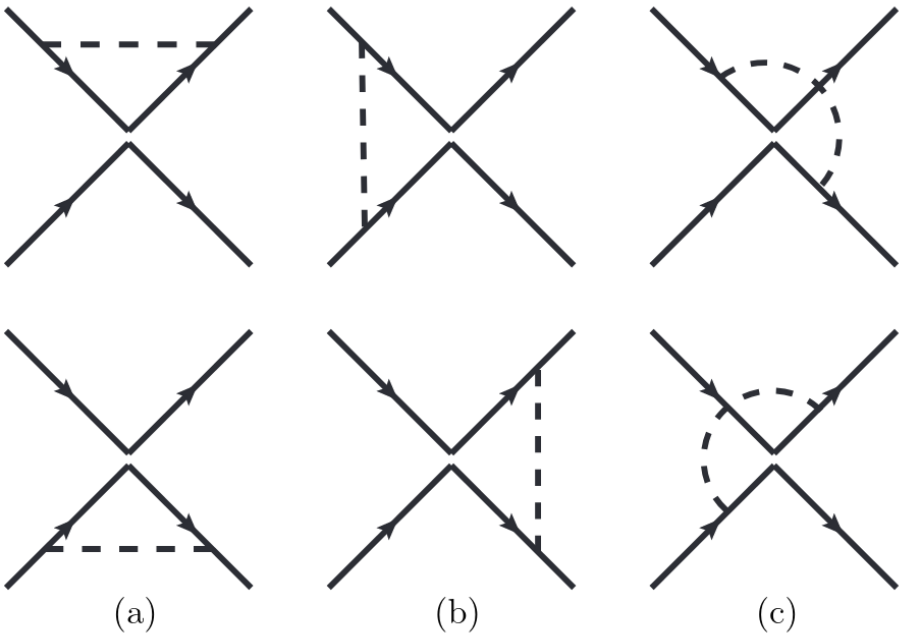


Figure 2: Contribution to the $\psi\psi \to \psi\psi$ amplitude in the EFT (crossed diagrams are not explicitly shown).

Before absorbing this divergence in the corresponding counterterm we consider the divergence corresponding to the kinetic term. In this case, since the contribution is proportional to p^2, we cannot set to zero external momenta, but compute up to this order in the corresponding expansion. Let's do the calculation (note that we can neglect the masses, since we are only interested in the leading contribution to the kinetic term)

$$= (-\mathrm{i}\eta)^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{i}}{k^2} \frac{\mathrm{i}(\not{k} + \not{p})}{(k+p)^2} = \frac{\mathrm{i}}{16\pi^2\varepsilon} \left(\frac{\eta^2}{2}\right) \not{p} + \text{ finite.} \quad (130)$$

The original plus divergent Lagrangian fixes the counterterms and, therefore, the beta functions.

$$\begin{aligned} \mathcal{L} &= \left(1 + \frac{\eta^2}{2} \frac{1}{16\pi^2\varepsilon}\right) \bar{\psi} \mathrm{i} \not{\partial} \psi + \frac{C}{2} \left(1 - 2\eta^2 \frac{1}{16\pi^2\varepsilon}\right) (\bar{\psi}\psi)^2 + \dots \\ &\to \bar{\psi} \mathrm{i} \not{\partial} \psi + \frac{C}{2} \left(1 - 3\eta^2 \frac{1}{16\pi^2\varepsilon}\right) (\bar{\psi}\psi)^2 + \dots \ , \end{aligned} \quad (131)$$

where, in the second line, we have canonically normalised the divergent Lagrangian so that we do not have to deal explicitly with wave-function renormalisation constants. The field redefinition needed to canonically normalise is

$$\psi \to \left(1 - \frac{\eta^2}{4} \frac{1}{16\pi^2\varepsilon}\right) \psi. \quad (132)$$

Once the divergent Lagrangian has been canonically normalised, the divergence in the four-fermion operator is parametrised by

$$C' = -\frac{3C\eta^2}{16\pi^2}, \tag{133}$$

and therefore the corresponding beta function reads,

$$\dot{C} = -2C' = \frac{6\eta^2 C}{16\pi^2} = \frac{3\eta^2 C}{8\pi^2}. \tag{134}$$

We can integrate this equation to fixed one loop order to obtain the leading logarithmic approximation

$$C(\mu_{\rm IR}) = C(\mu_{\rm UV}) + \frac{\dot{C}(\mu_{\rm UV})}{2}\ln\frac{\mu_{\rm IR}^2}{\mu_{\rm UV}^2} = C + \frac{3\eta^2 C}{16\pi^2}\ln\frac{\mu_{\rm IR}^2}{\mu_{\rm UV}^2}, \tag{135}$$

where we have used the fact that the beta function has been computed to one loop order and all the parameters on the right hand side of the last equality are evaluated at the UV scale $\mu_{\rm UV}$.

5.3.1 Resumming logarithms and RGE-improved perturbation theory

As we have seen in the previous section, the calculation of the beta function allows us to relate the WC at one scale with the corresponding value of the WC at another scale. RGEs can actually do more for us than just reproducing this relation to fix order in the loop expansion. We can use it to actually resum, to all loop orders, contributions that are logarithmically enhanced. Let's see an example of how this can be done. The one-loop beta function for the η coupling reads

$$\dot{\eta} = \frac{5}{16\pi^2}\eta^3, \tag{136}$$

which can be solved to give the value of the coupling at an arbitrary scale μ in terms of a UV scale, that we will denote here Λ, in the form

$$\eta^2(\mu) = \frac{\eta^2(\Lambda)}{1 - \frac{10}{16\pi^2}\eta^2(\Lambda)\ln\frac{\mu}{\Lambda}}. \tag{137}$$

We can now write a differential equation for C as follows

$$\frac{\mathrm{d}\ln C}{\mathrm{d}\ln\eta} = \frac{\mathrm{d}\ln C}{\mathrm{d}\ln\mu}\left(\frac{\mathrm{d}\ln\eta}{\mathrm{d}\ln\mu}\right)^{-1} = \frac{6}{5}, \tag{138}$$

which can be solved as follows

$$\begin{aligned} C(\mu) &= C(\Lambda)\left(\frac{\eta^2(\mu)}{\eta^2(\Lambda)}\right)^{\frac{3}{5}} = C(\Lambda)\left(1 - \frac{10}{16\pi^2}\eta^2(\Lambda)\ln\frac{\mu}{\Lambda}\right)^{-3/5} \\ &= \frac{\lambda^2}{M^2}\left(1 + \frac{3}{16\pi^2}\eta^2\ln\frac{\mu^2}{\Lambda^2} + \ldots\right). \end{aligned} \tag{139}$$

The important point is that by computing the one-loop beta functions, we have been able to include all loop order contributions of the form $(\alpha\ln)^n$, with n an arbitrary integer (n greater than 1 appears at higher than 1 loop) and $\alpha = \eta^2/(16\pi^2)$. Higher loop effects that are not included in this calculation are of the form $(\alpha)^m(\alpha\ln)^n$, which can be obtained by computing the corresponding beta functions at order $m+1$ loops. This constitutes the so-called RG-improved perturbation theory, in which the different orders are, symbolically, of the form

- Leading order (LO). Resums all terms of the form $(\alpha \ln)^n$ and cam be obtained by using the one-loop beta functions, in connection with the tree level matching conditions.
- Next-to-leading order (NLO). Resums all terms of the form $\alpha(\alpha \ln)^n$ and cam be obtained by using the two-loop beta functions, in connection with the one-loop matching conditions.
- . . . .

This RG-improved perturbation theory re-organises the perturbative expansion in such a way that it is well behaved when $\alpha \ll 1$ but there is a large hierarchy in scales such that $\alpha \ln \sim 1$. In this case, fixed order perturbation theory would have a very poor convergence behaviour and would not be predictive.

5.3.2 Operator mixing

In our previous example the divergence parametrised by $\mathcal{O}_{\psi^4}$ vanished if the operator itself was not present, *i.e.* $C' \propto C$. There are cases, however in which there is a divergence parametrised by a certain operator even in the absence of this operator. This gives rise to the so-called operator mixing (one operator induces another via renormalisation). Let us extend slightly our previous example, starting from the following effective Lagrangian

$$\mathcal{L}_{\text{EFT}} = -\frac{1}{2}\phi(\partial^2 + m_\phi^2)\phi + \bar{\psi}(\mathrm{i}\partial\!\!\!/ - m_\psi)\psi - \eta\bar{\psi}\psi\phi + \frac{C_s}{2}\bar{\psi}\psi\bar{\psi}\psi + \frac{C_v}{2}\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi + \frac{C_t}{2}\bar{\psi}\sigma^{\mu\nu}\psi\bar{\psi}\sigma_{\mu\nu}\psi, \tag{140}$$

where we have defined $\sigma^{\mu\nu} \equiv (\mathrm{i}/2)[\gamma^\mu, \gamma^\nu]$. This Lagrangian is the boundary condition for our RGEs. It should be understood as the Lagrangian written in terms of renormalised WCs at a certain renormalisation scale. The beta functions give the precise dependence on this renormalisation scale and allow us to compute, via the solution of the RGEs, the renormalised WCs at any other scale.

The beta functions can be computed following the procedure explained above. The result is

$$16\pi^2\beta_{m_\phi^2} = 4\eta^2(m_\phi^2 - 6m_\psi^2), \tag{141}$$
$$16\pi^2\beta_{m_\psi} = m_\psi[3\eta^2 + m_\psi^2(6C_s - 8C_v - 24C_t)], \tag{142}$$
$$16\pi^2\beta_\eta = \eta[5\eta^2 + m_\psi^2(18C_s - 24C_v - 72C_t)], \tag{143}$$
$$16\pi^2\beta_{C_s} = 6\eta^2 C_s, \tag{144}$$
$$16\pi^2\beta_{C_v} = 12\eta^2 C_t, \tag{145}$$
$$16\pi^2\beta_{C_t} = 2\eta^2(C_v + C_t). \tag{146}$$

Exercise 5.2. Reproduce the above results using `MatchmakerEFT` [23].

These results allow us to make several interesting points:

- We see operator mixing at work in β_{C_v} and β_{C_t}. If, at certain scale, we have $C_v \neq 0$ and $C_t = 0$, then the solution of the RGES at one loop will induce a non-vanishing value of C_t at any other scale. We say that $C_t = 0$ is not radiatively stable (at one-loop order). Interestingly, the opposite boundary condition, $C_v = 0$, $C_t \neq 0$, generates a non-zero value for C_v and induces one-loop corrections to C_t.

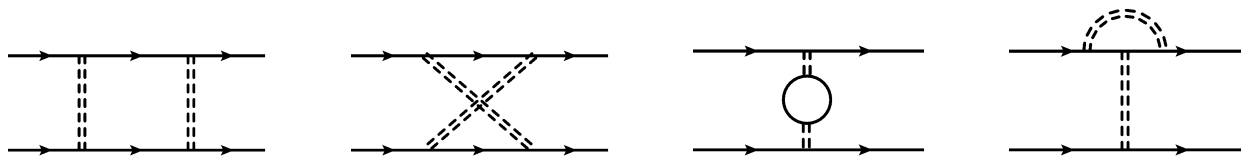


Figure 3: 1lPI contribution at one loop order to the $\psi\psi \to \psi\psi$ amplitude in the full theory. Only contributions proportional to λ^4 are considered. There are two diagrams of the last type and crossed terms have not been explicitly included.

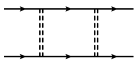
- Several beta functions are proportional to the coupling itself. In general this does not need to be the case, as we can see in the case of m_ϕ^2 or C_v but in many cases this is enforced by a symmetry reason. In particular, for the case of m_ψ chiral symmetry ensures that corrections to m_ψ have to be proportional to m_ψ itself, as it is the only parameter that breaks the symmetry. Similarly, η is the only parameter in the Lagrangian that breaks the $\phi \to -\phi$ symmetry.

- Dimensional analysis proves powerful here again. We can see, in the beta functions of η and m_ψ that higher-dimensional operators can contribute to (the beta function and, as we will see below to matching conditions) lower dimensional ones, multiplied by dimensionful scales, which can be (light) masses or cubic scalar interactions. Similarly, eventhough we don't see this in the example above, in general the product of operators of dimension higher than 4 renormalises operators of even higher dimension. For example, the insertion of two operators of dimension 6 will in general renormalise operators of dimension 8 and so on.

5.4 One-loop matching

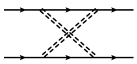
Let us now move on to discuss one-loop matching. We will do it, again, using our simple example, and restricting ourselves only to part of the matching. In particular, we will consider the dimension 6 contribution to $\mathcal{O}_{\psi^4}$ proportional to λ^4. We will follow the efficient matching procedure by simply computing the hard region contribution to the corresponding amplitudes in the full theory. The relevant diagrams, up to crossing of the final particles, are presented in Fig. 3. Let us compute the hard region contribution to each of the amplitudes in turn. At

mass-dimension 6 we can again neglect all external momenta and light masses.



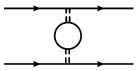
$$= (-\mathrm{i}\lambda)^4 \int_k \bar{u}_3 \frac{\mathrm{i}\not{k}}{k^2} u_1 \bar{u}_4 \frac{-\mathrm{i}\not{k}}{k^2} u_2 \frac{\mathrm{i}^2}{(k^2 - M^2)^2}$$

$$= -\lambda^4 \bar{u}_3 \gamma^\alpha u_1 \bar{u}_4 \gamma^\beta u_2 \int_k \frac{k^\alpha k^\beta}{k^4 (k^2 - M^2)}, \tag{147}$$

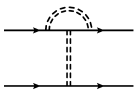


$$= (-\mathrm{i}\lambda)^4 \int_k \bar{u}_3 \frac{\mathrm{i}\not{k}}{k^2} u_1 \bar{u}_4 \frac{\mathrm{i}\not{k}}{k^2} u_2 \frac{\mathrm{i}^2}{(k^2 - M^2)^2}$$

$$= \lambda^4 \bar{u}_3 \gamma^\alpha u_1 \bar{u}_4 \gamma^\beta u_2 \int_k \frac{k^\alpha k^\beta}{k^4 (k^2 - M^2)}, \tag{148}$$



$$= 0 \text{ (scaleless)}, \tag{149}$$

$2 \times$ 

$$= 2 \times (-\mathrm{i}\lambda)^4 \frac{-\mathrm{i}}{M^2} \int_k \bar{u}_3 \frac{\mathrm{i}\not{k}}{k^2} \frac{\mathrm{i}\not{k}}{k^2} u_1 \bar{u}_4 u_2 \frac{\mathrm{i}}{k^2 - M^2}$$

$$= -\frac{2\lambda^4}{M^2} u_s \int_k \frac{1}{k^2 (k^2 - M^2)} = -\frac{2\lambda^4}{M^2} u_s \int_k \frac{1}{M^2} \left[\frac{1}{(k^2 - M^2)} - \frac{1}{k^2} \right]$$

$$= -\frac{\mathrm{i}}{16\pi^2} \frac{2\lambda^4}{M^2} u_s \left[\frac{1}{\bar{\varepsilon}} + 1 + \ln \frac{\mu^2}{M^2} \right], \tag{150}$$

where we have defined the following bispinor

$$u_s \equiv \bar{u}_3 u_1 \bar{u}_4 u_2. \tag{151}$$

Thus, we have

$$\mathrm{i}\mathcal{M}_\mathrm{F}\Big|^{\mathrm{ren}}_{\mathrm{hard}} = -\frac{2\mathrm{i}\lambda^4}{16\pi^2 M^2} \left(1 + \ln \frac{\mu^2}{M^2} \right) u_s. \tag{152}$$

As usual, we also need to canonically normalise the kinetic term so we also compute the contribution to the two point function (proportional to $\not{p}$). In order to get some practice, even if we do not need it for our calculation, we are going to also consider the contribution to

the mass term. The only contribution reads

$$
\begin{aligned}
\Big|_{\text{hard}} &= (-\mathrm{i}\lambda)^2 \int_k \frac{\mathrm{i}}{(k-p)^2 - M^2} \frac{\mathrm{i}(\not{k} + m)}{k^2 - m^2} \\
&= \lambda^2 \int_k \frac{1}{k^2 - M^2} \left[1 - \frac{p^2 - 2p\cdot k}{(k-p)^2 - M^2}\right] \frac{\not{k} + m}{k^2} \left[1 + \frac{m^2}{k^2} + \frac{m^4}{k^4} + \ldots\right] \\
&= \lambda^2 \int_k \frac{\not{k} + m}{k^2(k^2 - M^2)} \left[1 + \frac{2p\cdot k}{k^2 - M^2}\right] + \mathcal{O}(p^2, m^2) \\
&= \lambda^2 \int_k \left[\frac{m}{k^2(k^2 - M^2)} + \frac{2p\cdot k \not{k}}{k^2(k^2 - M^2)^2}\right] \\
&\to \lambda^2 \int_k \left[\frac{m}{k^2(k^2 - M^2)} + \frac{2}{D}\frac{\not{p}}{(k^2 - M^2)^2}\right] \\
&= \lambda^2 \int_k \left[\frac{m}{M^2}\left(\frac{1}{k^2 - M^2} - \frac{1}{k^2}\right) + \frac{2}{D}\frac{D-2}{2M^2}\frac{\not{p}}{k^2 - M^2}\right] \\
&= \frac{\mathrm{i}\lambda^2}{16\pi^2}\left(\frac{1}{\bar{\varepsilon}} + 1 + \ln\frac{\mu^2}{M^2}\right)\left[\frac{1}{2}\left(1 - \frac{\varepsilon}{2}\right)\not{p} + m\right] \\
&= \frac{\mathrm{i}\lambda^2}{16\pi^2}\left[\left(\frac{1}{2\bar{\varepsilon}} + \frac{1}{4} + \frac{1}{2}\ln\frac{\mu^2}{M^2}\right)\not{p} + \left(\frac{1}{\bar{\varepsilon}} + 1 + \ln\frac{\mu^2}{M^2}\right) m\right].
\end{aligned} \tag{153}
$$

As promised, the hard contributions are local, and can be parametrised by the following Lagrangian (neglecting divergent terms that are, either MS-barred away if they are of UV origin or cancelled in the matching if they are IR)

$$
\mathcal{L}_{\text{EFT}} = \bar{\psi}[K\mathrm{i}\partial\!\!\!/ - C_m]\psi + \frac{C_{\psi^4}}{2}\bar{\psi}\psi\bar{\psi}\psi + \ldots, \tag{154}
$$

where

$$
K = 1 + \frac{\lambda^2}{16\pi^2}\left(\frac{1}{4} + \frac{1}{2}\ln\frac{\mu^2}{M^2}\right), \tag{155}
$$

$$
C_m = m\left[1 + \frac{\lambda^2}{16\pi^2}\left(1 + \ln\frac{\mu^2}{M^2}\right)\right], \tag{156}
$$

$$
C_{\psi^4} = \frac{\lambda^2}{M^2}\left[1 - \frac{\lambda^2}{16\pi^2}\left(2 + 2\ln\frac{\mu^2}{M^2}\right)\right]. \tag{157}
$$

Canonically normalising via

$$
\psi \to 1 - \frac{\lambda^2}{16\pi^2}\left(\frac{1}{8} + \frac{1}{4}\ln\frac{\mu^2}{M^2}\right)\psi, \tag{158}
$$

we get

$$
K \to 1, \tag{159}
$$

$$
C_m \to m\left[1 + \frac{\lambda^2}{16\pi^2}\left(\frac{3}{4} + \frac{1}{2}\ln\frac{\mu^2}{M^2}\right)\right], \tag{160}
$$

$$
C_{\psi^4} \to \frac{\lambda^2}{M^2}\left[1 - \frac{\lambda^2}{16\pi^2}\left(\frac{5}{2} + 3\ln\frac{\mu^2}{M^2}\right)\right]. \tag{161}
$$

Several comments are in order.

- Note that the one-loop correction to the fermion mass is proportional to the fermion mass itself. The reason is that, in the limit $m = 0$ there is a higher chiral symmetry, under which the left and right handed components of ψ transform differently. Since the fermion mass is the only parameter than breaks this symmetry, all corrections must be proportional to the mass itself. We say the fermion mass has a chiral protection. This chiral protection is behind the solution of the hierarchy problem in supersymmetric theories (together with a soft breaking of supersymmetry).

- Our results for the WCs depend explicitly on μ, reflecting the fact that they are MSbar-renormalised WCs (defined at the scale μ). As usual, the renormalised scale, μ, is arbitrary, but it is usually chosen at or near the mass threshold we are integrating out (M in our case) to minimise the logs appearing in the expression. We can however keep the explicit dependence on μ. In the next section we discuss how this μ dependence is compatible with the beta function of the corresponding WCs, once all implicit dependences are included.

5.4.1 RGEs and μ-dependence on the matching results

We have obtained in our previous calculation that, up to one loop order,

$$C_{\psi^4} = \frac{\lambda^2}{M^2}\left[1 - \frac{\lambda^2}{16\pi^2}\left(\frac{5}{2} + 3\ln\frac{\mu^2}{M^2}\right)\right], \tag{162}$$

where all parameters in this expression are MSbar-renormalised parameters (up to one-loop order), defined at the renormalisation scale μ. In the previous section, we saw that the beta function for this WC is

$$\dot{C}_{\psi^4} = \frac{\mathrm{d}C_{\psi^4}}{\ln\mu} = \frac{6}{16\pi^2}\frac{\eta^2\lambda^2}{M^2}. \tag{163}$$

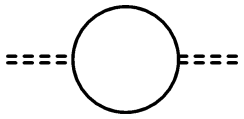
However, the term in front of $\ln(\mu/M)$ in Eq. (162) is

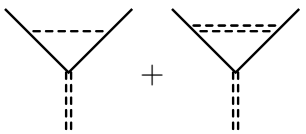
$$-\frac{6}{16\pi^2}\frac{\lambda^4}{M^2}, \tag{164}$$

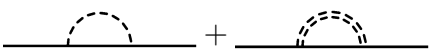
which does not agree with the expectation from the EFT. The reason is that we have not computed the full dependence on the renormalisation scale of the WC in Eq. (162). The parameters λ and M are, as emphasized, renormalised parameters that also depend on μ. The full dependence is thus

$$\dot{C}_{\psi^4} = \frac{2\lambda\dot{\lambda}}{M^2} - \frac{\lambda^2(\dot{M^2})}{M^4} - \frac{6}{16\pi^2}\frac{\lambda^4}{M^2}. \tag{165}$$

We need to compute the one-loop beta functions of λ and M^2 in the full theory. Let us give the result for the relevant diagrams


$$= \frac{2\mathrm{i}\lambda^2}{16\pi^2\varepsilon}(p^2 - 6m^2) + \ldots, \tag{166}$$


$$= \frac{\mathrm{i}\lambda}{16\pi^2\varepsilon}(\eta^2 + \lambda^2) + \ldots, \tag{167}$$


$$= \frac{\mathrm{i}\not{p}}{16\pi^2\varepsilon}\frac{1}{2}(\eta^2 + \lambda^2) + \ldots, \tag{168}$$

The original plus divergent Lagrangian therefore reads,

$$\mathcal{L} = \left(1 + \frac{1}{16\pi^2\varepsilon}\frac{\eta^2+\lambda^2}{2}\right)\bar{\psi}\mathrm{i}\partial\!\!\!/\psi + \left(1+\frac{2\lambda^2}{16\pi^2\varepsilon}\right)\frac{1}{2}(\partial_\mu\Phi)^2 - \frac{1}{2}M^2\Phi^2 - \lambda\left(1-\frac{\eta^2+\lambda^2}{16\pi^2\varepsilon}\right)\bar{\psi}\psi\Phi + \ldots,$$
$$\to \bar{\psi}\mathrm{i}\partial\!\!\!/\psi + \frac{1}{2}(\partial_\mu\Phi)^2 - \left(1-\frac{2\lambda^2}{16\pi^2\varepsilon}\right)\frac{1}{2}M^2\Phi^2 - \lambda\left(1-\frac{1}{16\pi^2\varepsilon}\frac{3\eta^2+5\lambda^2}{2}\right)\bar{\psi}\psi\Phi + \ldots, \quad (169)$$

where in the second line we have canonically normalised. From this we can read off the beta functions for λ and M^2,

$$\dot{M^2} = -2(M^2)' = \frac{4\lambda^2}{16\pi^2}M^2, \quad (170)$$

$$\dot{\lambda} = -2\lambda' = \frac{(3\eta^2+5\lambda^2)\lambda}{16\pi^2}. \quad (171)$$

We can now put everything together to obtain

$$\begin{aligned}\dot{C}_{\psi^4} &= \frac{2\lambda\dot{\lambda}}{M^2} - \frac{\lambda^2(\dot{M^2})}{M^4} - \frac{6}{16\pi^2}\frac{\lambda^4}{M^2} \\ &= \frac{1}{16\pi^2}\frac{\lambda^2}{M^2}\left[2(3\eta^2+5\lambda^2) - 4\lambda^2 - 6\lambda^2\right] = \frac{6}{16\pi^2}\frac{\lambda^2\eta^2}{M^2},\end{aligned} \quad (172)$$

which is fully consistent with the beta function as computed in the EFT itself, Eq. (163).

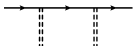
5.4.2 One-loop matching: complete calculation in the full theory and in the EFT

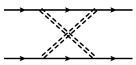
In the previous sections we have computed the one-loop matching conditions directly from the hard region contribution of the corresponding amplitudes in the full theory. One can, of course, take the longer route of computing the full amplitude in both the full theory and in the EFT and subtract the two. This procedure involves computing the complete one-loop amplitude in the full theory and then (after integration) expand in the heavy scaler. We then have to compute the complete one-loop amplitude in the EFT and subtract the two. Why would one want to go through the trouble of matching onto the EFT if we have already computed the full theory amplitude? Usually we do not, and this is why we use the expansion in the hard region in the firs place, but sometimes we have to, because the presence of disparate scales requires us to resum the corresponding logarithms using the EFT RGEs. We will briefly mention an example below. Note that, even if this resummation was not needed, it still can make sense to expand the full theory amplitude and match it to the EFT. The reason is that, in order to compute the matching, there is only a finite number of amplitudes we need to compute and, once we have the EFT with the matched WCs, we can compute any other physical process in the simpler context of the EFT itself.

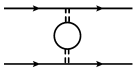
In any case, in this section we are going to give the results of this process for the one-loop matching of the $\mathcal{O}(\lambda^4)$ contribution to C_{ψ^4}. We do this for pedagogical reasons, to show that our expectation in terms of cancellation of non-local terms in the light scales cancel in the matching are indeed fulfilled. Let us define the following two bi-spinors

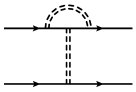
$$u_s \equiv \bar{u}_3 u_1 \bar{u}_4 u_2, \quad u_v \equiv \bar{u}_3\gamma^\mu u_1 \bar{u}_4\gamma_\mu u_2. \quad (173)$$

As usual we leave the crossed $(3 \leftrightarrow 4)$ contribution implicit. We obtain


$$= \frac{\mathrm{i}\lambda^4}{16\pi^2 M^2}\left[u_s \frac{m^2}{M^2}\left(\ln\frac{M^2}{m^2}-2\right)+u_v\left(\frac{1}{4}+\frac{1}{4}\frac{m^2}{M^2}(3-2\ln\frac{M^2}{m^2})\right)\right]+\dots\,, \quad (174)$$


$$= \frac{\mathrm{i}\lambda^4}{16\pi^2 M^2}\left[u_s \frac{m^2}{M^2}\left(\ln\frac{M^2}{m^2}-2\right)-u_v\left(\frac{1}{4}+\frac{1}{4}\frac{m^2}{M^2}(3-2\ln\frac{M^2}{m^2})\right)\right]+\dots\,, \quad (175)$$

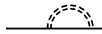

$$= \frac{\mathrm{i}\lambda^4}{16\pi^2 M^2}\left[u_s\left(-4\frac{m^2}{M^2}\right)\left(\frac{3}{\bar{\varepsilon}}+1+3\ln\frac{\mu^2}{m^2}\right)\right]+\dots\,, \quad (176)$$

$2\times$ 
$$= \frac{\mathrm{i}\lambda^4}{16\pi^2 M^2}\left[u_s\left(-\frac{2}{\bar{\varepsilon}}-2-2\ln\frac{\mu^2}{m^2}+\frac{m^2}{M^2}\left(6\ln\frac{M^2}{m^2}-4\right)\right)\right]+\dots\,. \quad (177)$$

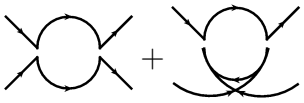
Adding everything together we get

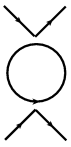
$$\mathrm{i}\mathcal{M}_F = \frac{\mathrm{i}}{16\pi^2}\frac{2\lambda^4}{M^2}u_s\left[-\frac{1}{\bar{\varepsilon}}-1-\ln\frac{\mu^2}{M^2}+\frac{m^2}{M^2}\left(-\frac{6}{\bar{\varepsilon}}-6\ln\frac{\mu^2}{m^2}-6+4\ln\frac{M^2}{m^2}\right)\right]+\dots\,. \quad (178)$$

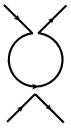
Similarly, the two point function receives a contribution to the kinetic term that reads


$$= \frac{\mathrm{i}}{16\pi^2}\frac{\lambda^2}{2}\not{p}\left(\frac{1}{\bar{\varepsilon}}+\frac{1}{2}+\ln\frac{\mu^2}{M^2}\right)+\dots\,. \quad (179)$$

Let us now move to the EFT side. The corresponding diagrams and their result are


$$= \frac{2\mathrm{i}C_{\psi^4}^2 m^2}{16\pi^2}u_s\left[\frac{1}{\bar{\varepsilon}}+\ln\frac{\mu^2}{m^2}\right]+\dots\,, \quad (180)$$

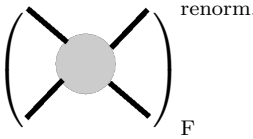

$$= -\frac{4\mathrm{i}C_{\psi^4}^2 m^2}{16\pi^2}u_s\left[\frac{3}{\bar{\varepsilon}}+1+3\ln\frac{\mu^2}{m^2}\right]+\dots\,, \quad (181)$$

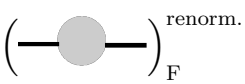
$2\times$ 
$$= \frac{2\mathrm{i}C_{\psi^4}^2 m^2}{16\pi^2}u_s\left[\frac{3}{\bar{\varepsilon}}+1+3\ln\frac{\mu^2}{m^2}\right]+\dots\,. \quad (182)$$

Adding everything together we get

$$\mathrm{i}\mathcal{M}_E = -\frac{2\mathrm{i}\lambda^4}{16\pi^2}\frac{m^2}{M^4}u_s\left[\frac{2}{\bar{\varepsilon}}+1+2\ln\frac{\mu^2}{m^2}\right]+\dots\,. \quad (183)$$

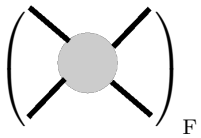
There is no contribution to the fermion kinetic term in the EFT at the required order. Renormalising all the relevant amplitudes (all divergences have a UV origin, as IR ones are regulated by the external momenta or masses) and subtracting, we obtain


$$= \frac{\mathrm{i}}{16\pi^2}u_s\frac{2\lambda^4}{M^2}\left[-1-\ln\frac{\mu^2}{M^2}+\frac{m^2}{M^2}\left(-5-4\ln\frac{\mu^2}{M^2}\right)\right]+\dots\,, \quad (184)$$

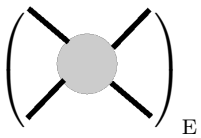

$$= \frac{\mathrm{i}}{16\pi^2}u_s\not{p}\frac{\lambda^2}{2}\left[\frac{1}{2}+\ln\frac{\mu^2}{M^2}\right]+\dots\,, \quad (185)$$

which completely agree, at dimension 6, with or previous alculation, Eqs. (155-157).

As usual, several comments are in order. Let us look a bit more in detail at the $\psi\psi \to \psi\psi$ amplitudes in the full and effective theories,



$$= \frac{\mathrm{i}}{16\pi^2}\frac{2\lambda^4}{M^2}u_s\left[-\frac{1}{\bar{\varepsilon}} - 1 - \ln\frac{\mu^2}{M^2} + \frac{m^2}{M^2}\left(-\frac{6}{\bar{\varepsilon}} - 6\ln\frac{\mu^2}{m^2} - 6 + 4\ln\frac{M^2}{m^2}\right)\right] + \dots \,, \tag{186}$$



$$- \frac{2\mathrm{i}\lambda^4}{16\pi^2}\frac{m^2}{M^4}u_s\left[\frac{2}{\bar{\varepsilon}} + 1 + 2\ln\frac{\mu^2}{m^2}\right] + \dots \,. \tag{187}$$

Now our comments:

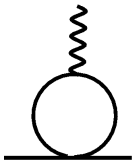
- Note that in the EFT we need two insertions of Op_{ψ^4} so, since we are using a mass-independent renormalisation scheme, we know directly that the contribution is going to be dimension 8.
- UV divergences are different in the full theory and in the EFT, both at mass-dimension 6 and mass-dimension 8. This is expected, as both theories are different in the UV and they renormalise differently.
- At mass-dimension 8 we find in both amplitudes a non-local dependence on the light mass m. This is an IR effect, that should be identical in both theories and indeed we see that the dependence on $\ln m^2$ is identical in both and cancels in the difference, so that the WCs are local in all the IR (light) scales.

5.4.3 Mass-independent vs cut-off renormalisation schemes

We have seen how a mass-independent renormalisation scheme allows us to use a purely dimensional power counting method. Indeed, if we used a different regulator that introduced scales in the integration process, we would break the perturbative expansion based on the mass dimension of the different operators. This example has been taken from [24]. Consider the following effective Lagrangian

$$\mathcal{L} = -\frac{4G_F}{\sqrt{2}}V_{ui}V_{ui}^*(\bar{u}\gamma^\mu P_L q_i)(\bar{q}_i\gamma_\mu P_L u) + \dots \tag{188}$$

This term induces a one-loop contribution to the $Zu_L\bar{u}_L$ coupling of the form



$$\sim I_6 \sim \frac{1}{M_W^2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2} \sim \frac{\Lambda^2}{M_W^2} \sim \mathcal{O}(1), \tag{189}$$

where in the last identity we have used the fact that the physical cut-off should be of the same order as the W mass. A similar operator of mass-dimension 8 (with 2 extra derivatives) would contribute

$$I_8 \sim \frac{1}{M_W^2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2}\frac{k^2}{M_W^2} \sim \frac{\Lambda^4}{M_W^4} \sim \mathcal{O}(1). \tag{190}$$

Thus, we see that, at the loop level, a regularisation scheme that is not mass independent can mess up with the operator dimensions and the contribution of higher-dimensional operators is not suppressed with respect to the one of lower-dimensional ones. The equivalent results in a mass-independent scheme would be

$$I_6 \sim \frac{1}{M_W^2}\int_k \frac{1}{k^2} \sim \frac{m^2}{M_W^2}\left(a + b\ln\frac{\mu^2}{M_W^2}\right) \ll 1, \tag{191}$$

$$I_8 \sim \frac{1}{M_W^4}\int_k \sim \frac{m^4}{M_W^4}\left(a' + b'\ln\frac{\mu^2}{M_W^2}\right) \ll I_6 \ll 1. \tag{192}$$

It is therefore clear that one should use in general a mass-independent regularisation scheme when using EFTs.

The problem with mass-independent regularisation schemes is that they do not decouple. In particular, the beta functions are mass independent and therefore heavy masses contribute to the running just as much as light ones. Indeed, in QED we have, for the one-loop beta function of the coupling constant

$$\beta^{\overline{\text{MS}}}(e) = \frac{e^3}{12\pi^2}, \tag{193}$$

$$\beta^{\text{mom.}}(e) = \frac{e^3}{2\pi^2}\int_0^1 \mathrm{d}x\, x(1-x)\frac{\mu^2 x(1-x)}{M^2 + \mu^2 x(1-x)} \sim \begin{cases} \frac{e^3}{12\pi^2}, & M \ll \mu, \\ \frac{e^3}{12\pi^2}\frac{\mu^2}{5M^2}, & \mu \ll M. \end{cases}, \tag{194}$$

where mom. stands for momentum substraction at a scale μ, which is a mass-dependent regularisation scheme and indeed decouples.

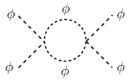
The solution is however at our hand. We can, and should, use a mass-independent scheme and everytime we cross a physical threshold (the mass of a particle) we decouple it by hand by matching the theory to an EFT in which that particle is not present.

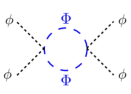
6 When EFT is the only way

Even if we are working with renormalisable theories in which we can compute any physical observable at any loop order by renormalising our theory with a finite number of counterterms, we might be forced to use EFTs to reliably compute certain observables. This example is taken from [25]. Consider the following renormalisable Lagrangian

$$\mathcal{L} = -\frac{1}{2}\phi(\partial^2 + m^2)\phi - \frac{1}{2}\Phi(\partial^2 + M^2)\Phi - \frac{\eta}{4!}\phi^4 - \frac{\kappa}{4}\phi^2\Phi^2. \tag{195}$$

We want to compute $\phi\phi \to \phi\phi$ scattering at threshold. We will give only the results here, full details of the calculation can be found in [25]. The contribution is

$3 \times$  $= \frac{3\mathrm{i}}{32\pi^2}\eta^2\left(\frac{1}{\bar{\varepsilon}} + \frac{2}{3} + \ln\frac{\mu^2}{m^2}\right),$ (196)

$3 \times$  $= \frac{3\mathrm{i}}{32\pi^2}\kappa^2\left(\frac{1}{\bar{\varepsilon}} + \ln\frac{\mu^2}{m^2}\right).$ (197)

Thus, the total, up to one-loop order, amplitude reads, after renormalisation

$$i\mathcal{M}_F = -\mathrm{i}\eta + \frac{\mathrm{i}}{16\pi^2}\left[\eta^2\left(1+\frac{3}{2}\ln\frac{\mu^2}{m^2}\right)+\kappa^2\frac{3}{2}\ln\frac{\mu^2}{M^2}\right]. \tag{198}$$

The important point is that, if $m \ll M$, there is no choice of μ that makes all logarithms small. Thus, even if

$$\frac{\eta^2}{16\pi^2} \ll 1, \quad \frac{\kappa^2}{16\pi^2} \ll 1, \tag{199}$$

if

$$\frac{\eta^2}{16\pi^2}\ln\frac{M^2}{m^2} \sim \frac{\kappa^2}{16\pi^2}\ln\frac{M^2}{m^2} \sim \mathcal{O}(1), \tag{200}$$

then the perturbative expansion breaks down (if only one of the terms is order one then in this particular case we could choose μ to make the relevant large log vanishing but the problem would arise at higher loops and in general theories is present already at one loop order).

Let us see how, using the EFT approach, we can resum the large logs to all orders and get a result that is well behaved perturbatively. The relevant effective Lagrangian reads

$$\mathcal{L}_{\rm EFT} = -\frac{1}{2}\phi(\partial^2+m^2)\phi - \frac{C}{4!}\phi^4 + \dots. \tag{201}$$

The matching, up to one loop gives (no correction to the kinetic term up to this order), reads

$$C \equiv C(\mu_M) = \eta(\mu_M) - \frac{3}{2}\frac{1}{16\pi^2}\kappa^2(\mu_M)\ln\frac{\mu_M^2}{M^2} \equiv \eta - \frac{3}{2}\frac{1}{16\pi^2}\kappa^2\ln\frac{\mu_M^2}{M^2}, \tag{202}$$

where we have explicitly shown that all our WCs are renormalised at the matching scale μ_M and implicitly denoted the renormalised WCs without an explicit scale dependence as evaluated at such scale. We can choose $\mu_M \sim M$ to ensure that the log is small. We can now compute C at any other lower scale by using its beta function,

$$\dot{C} = 3\frac{C^2}{16\pi^2} \Rightarrow C(\mu_L) = \frac{C}{1-\frac{3}{2}\frac{1}{16\pi^2}C\ln\frac{\mu_L^2}{\mu_M^2}}, \tag{203}$$

where by solving the RGE exacly we have resummed all powers of $C\times\ln$. Computing now the amplitude in the EFT at the scale μ_L we get

$$\mathrm{i}\mathcal{M}_E = -\mathrm{i}C(\mu_L) + \frac{\mathrm{i}}{16\pi^2}C^2(\mu_L)\left(1+\frac{3}{2}\ln\frac{\mu_L^2}{m^2}\right), \tag{204}$$

in which the log is also small provided we choose $\mu_L \sim m$. We have used the EFT approach to split a two-scale problem into two single-scale calculations, connected by the RGE in the EFT that allows us to sum the large logs to all orders in perturbation theory. Indeed, if we expand our result in the EFT we get back the fixed order result we computed before. Let us see it. The expanded (to one-loop) solution to the RGE reads

$$\begin{aligned} C(\mu_L) &= \frac{C}{1-\frac{3}{32\pi^2}C\log\frac{\mu_L^2}{\mu_M^2}} = C + \frac{3}{32\pi^2}C^2\log\frac{\mu_L^2}{\mu_M^2} + \dots \\ &= \eta + \frac{3}{2}\frac{1}{16\pi^2}\left[\eta^2\log\frac{\mu_L^2}{\mu_M^2} - \kappa^2\log\frac{\mu_M^2}{M^2}\right] + \dots, \end{aligned} \tag{205}$$

where in the last line we have replaced C for its value at the matching scale up to one-loop order. We can replace this in the expression for the EFT amplitude to obtain

$$\begin{aligned}
\mathrm{i}\mathcal{M}_E &= -\mathrm{i}C(\mu_L) + \frac{\mathrm{i}}{16\pi^2}C(\mu_L)^2\left(1+\frac{3}{2}\ln\frac{\mu_L^2}{m^2}\right)\\
&= -\mathrm{i}\eta - \mathrm{i}\frac{3}{2}\frac{1}{16\pi^2}\left[\eta^2\ln\frac{\mu_L^2}{\mu_M^2} - \kappa^2\ln\frac{\mu_M^2}{M^2}\right]\\
&+ \frac{\mathrm{i}}{16\pi^2}\eta^2\left(1+\frac{3}{2}\ln\frac{\mu_L^2}{m^2}\right)\\
&= -\mathrm{i}\eta + \frac{\mathrm{i}}{16\pi^2}\left[\eta^2\left(1+\frac{3}{2}\ln\frac{\mu_M^2}{m^2}\right) + \kappa^2\frac{3}{2}\ln\frac{\mu_M^2}{M^2}\right],
\end{aligned} \tag{206}$$

which agrees with our one-loop fixed order calculation.

Recall what we discussed about RGE resummation of logs and RG-improved perturbation theory. Let us consider the case in which

$$\frac{\eta^2}{16\pi^2}\sim\frac{\kappa^2}{16\pi^2}\ll 1,\ \text{but}\ \frac{\eta^2}{16\pi^2}\ln\frac{m^2}{M^2}\sim\mathcal{O}(1). \tag{207}$$

We would then get the leading order approximation (LO) by matching at tee-level, running at one loop and computing the amplitude in the EFT at tree level, so that the amplitude reads

$$\mathrm{i}\mathcal{M}_{\mathrm{E}}^{\mathrm{LO}} = -\mathrm{i}C(\mu_L) = -\mathrm{i}\frac{\eta}{1-\frac{3}{32\pi^2}\eta\ln\frac{\mu_L^2}{\mu_M^2}}. \tag{208}$$

NLO would involve one loop matching at the scale μ_M, which we have actually already done in Eq. (202), two loop running from μ_M down to μ_L (that we have not done) and one loop amplitude in the EFT, again done up to this order in Eq. (204). Let us, without actually computing the two loop beta functions, show how the full NLO calculation works. Let us assume [6] that the two loop running of C_4 can be written, up to two loop order, as follows

$$\dot{C} = \frac{b_0}{16\pi^2}C^2 + \frac{b_1}{(16\pi^2)^2}C^3, \tag{209}$$

where the beta function, up to two loops reads [7]

$$b_0 = 3,\quad b_1 = -\frac{17}{3}. \tag{210}$$

Since there are several couplings involved in the calculation, we have explicitly included a factor of $1/(16\pi^2)$ to keep track of the loop order. The large log condition therefore reads

$$\hat{L}\equiv\frac{1}{16\pi^2}L\equiv\frac{1}{16\pi^2}\ln\frac{\mu_M^2}{\mu_L^2}\sim\mathcal{O}(1),\ \text{large log condition}, \tag{211}$$

where we have denoted the large log by L. We can iteratively solve the corresponding RGE to find, up to two loop order

$$C(\mu_L) = \frac{C}{1-\frac{b_0}{2}C\hat{L}-\frac{b_1}{2}C^2\frac{\hat{L}}{16\pi^2}}, \tag{212}$$

[6] In fact, this is exactly true if we neglect higher-order effects, coming from dimension 6 operators into dimension 4 ones. This is an order m^2/Λ^2 that we neglect here.

[7] We thank Javier Fuentes-Martín for providing the two loop beta function.

where we see that the correction proportional to b_0, the LO one, is order one, whereas the one proportional to b_1 has a one-loop $1/16\pi^2$ suppression. Let us now put everything together. The one-loop matching gives

$$C = \eta - \frac{3\kappa^2 l_M}{2}\frac{1}{16\pi^2}, \tag{213}$$

where we remind the reader that renormalised quantities for which we do not explicitly show the renormalisation scale are evaluated at the matching scale $\mu = \mu_M$ and we have defined

$$l_M \equiv \ln\frac{\mu_M^2}{M^2}. \tag{214}$$

This is a small logarithm (provided we choose $\mu_M \sim M$) and therefore the term proportional to it has a genuine one-loop $1/16\pi^2$ suppression with respect to the first term. Replacing this expression into the solution of the two-loop RGE, Eq. (212) we have

$$\begin{aligned} C(\mu_L) &= \frac{\eta - \frac{3\kappa^2 l_M}{2}\frac{1}{16\pi^2}}{1 - \frac{b_0}{2}\left(\eta - \frac{3\kappa^2 l_M}{2}\frac{1}{16\pi^2}\right)\hat{L} - \frac{b_1}{2}\frac{\eta^2}{16\pi^2}\hat{L}} \\ &= \frac{\eta - \frac{3\kappa^2 l_M}{2}\frac{1}{16\pi^2}}{1 - \frac{b_0}{2}\eta\hat{L} + \left(\frac{b_0}{2}\frac{3\kappa^2 l_M}{2} - \frac{b_1}{2}\eta^2\right)\frac{1}{16\pi^2}\hat{L}}, \\ &= \frac{\eta}{1 - \frac{b_0}{2}\eta\hat{L}} - \frac{\left(1 - \frac{b_0}{2}\eta\hat{L}\right)\frac{3\kappa^2 l_M}{2} + \eta\left(\frac{b_0}{2}\frac{3\kappa^2 l_M}{2} - \frac{b_1}{2}\eta^2\right)\hat{L}}{\left(1 - \frac{b_0}{2}\eta\hat{L}\right)^2}\frac{1}{16\pi^2}. \end{aligned} \tag{215}$$

Inserting this in the expression for the EFT amplitude we then have

$$\begin{aligned} \mathcal{M}_{\rm E}^{\rm NLO} &= -C(\mu_L) + C^2(\mu_L)\left(1 + \frac{3}{2}l_m\right)\frac{1}{16\pi^2} \\ &= -\frac{\eta}{1 - \frac{b_0}{2}\eta\hat{L}} \\ &+ \left\{\frac{\left(1 - \frac{b_0}{2}\eta\hat{L}\right)\frac{3\kappa^2 l_M}{2} + \eta\left(\frac{b_0}{2}\frac{3\kappa^2 l_M}{2} - \frac{b_1}{2}\eta^2\right)\hat{L}}{\left(1 - \frac{b_0}{2}\eta\hat{L}\right)^2} + \left(\frac{\eta}{1 - \frac{b_0}{2}\eta\hat{L}}\right)^2\left(1 + \frac{3}{2}l_m\right)\right\}\frac{1}{16\pi^2}, \end{aligned} \tag{216}$$

where we have defined

$$l_m \equiv \ln\frac{\mu_L^2}{m^2}, \tag{217}$$

which, again, is not a large logarithm. The NLO correction, in curly brackets, has a genuine $1/16\pi^2$ suppression.

Acknowledgments

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References

[1] H. Lehmann, K. Symanzik, and W. Zimmermann. On the formulation of quantized field theories. *Nuovo Cim.*, 1:205–225, 1955.

[2] G. F. Giudice, C. Grojean, A. Pomarol, and R. Rattazzi. The Strongly-Interacting Light Higgs. *JHEP*, 06:045, 2007.

[3] Konstantin G. Chetyrkin, Mikolaj Misiak, and Manfred Munz. Beta functions and anomalous dimensions up to three loops. *Nucl. Phys. B*, 518:473–494, 1998.

[4] Florian Herren and Anders Eller Thomsen. On ambiguities and divergences in perturbative renormalization group functions. *JHEP*, 06:116, 2021.

[5] Christopher Arzt. Reduced effective Lagrangians. *Phys. Lett. B*, 342:189–195, 1995.

[6] J. C. Criado and M. Pérez-Victoria. Field redefinitions in effective theories at higher orders. *JHEP*, 03:038, 2019.

[7] Javier Fuentes-Martín, Matthias König, Julie Pagès, Anders Eller Thomsen, and Felix Wilsch. Evanescent Operators in One-Loop Matching Computations. 11 2022.

[8] B. Grzadkowski, M. Iskrzynski, M. Misiak, and J. Rosiek. Dimension-Six Terms in the Standard Model Lagrangian. *JHEP*, 10:085, 2010.

[9] L. F. Abbott. The Background Field Method Beyond One Loop. *Nucl. Phys. B*, 185:189–203, 1981.

[10] Anders Eller Thomsen. A Partially Fixed Background Field Gauge. 4 2024.

[11] W. Buchmuller and D. Wyler. Effective Lagrangian Analysis of New Interactions and Flavor Conservation. *Nucl. Phys. B*, 268:621–653, 1986.

[12] Valerio Gherardi, David Marzocca, and Elena Venturini. Matching scalar leptoquarks to the SMEFT at one loop. *JHEP*, 07:225, 2020. [Erratum: JHEP 01, 006 (2021)].

[13] Juan Carlos Criado. BasisGen: automatic generation of operator bases. *Eur. Phys. J. C*, 79(3):256, 2019.

[14] Renato M. Fonseca. The Sym2Int program: going from symmetries to interactions. *J. Phys. Conf. Ser.*, 873(1):012045, 2017.

[15] Renato M. Fonseca. Enumerating the operators of an effective field theory. *Phys. Rev. D*, 101(3):035040, 2020.

[16] Landon Lehman. Extending the Standard Model Effective Field Theory with the Complete Set of Dimension-7 Operators. *Phys. Rev. D*, 90(12):125023, 2014.

[17] Brian Henning, Xiaochuan Lu, Tom Melia, and Hitoshi Murayama. 2, 84, 30, 993, 560, 15456, 11962, 261485, ...: Higher dimension operators in the SM EFT. *JHEP*, 08:016, 2017. [Erratum: JHEP 09, 019 (2019)].

[18] Yi Liao and Xiao-Dong Ma. Operators up to Dimension Seven in Standard Model Effective Field Theory Extended with Sterile Neutrinos. *Phys. Rev. D*, 96(1):015012, 2017.

[19] Hao-Lin Li, Zhe Ren, Jing Shu, Ming-Lei Xiao, Jiang-Hao Yu, and Yu-Hui Zheng. Complete set of dimension-eight operators in the standard model effective field theory. *Phys. Rev. D*, 104(1):015026, 2021.

[20] Christopher W. Murphy. Dimension-8 operators in the Standard Model Effective Field Theory. *JHEP*, 10:174, 2020.

[21] Hao-Lin Li, Zhe Ren, Ming-Lei Xiao, Jiang-Hao Yu, and Yu-Hui Zheng. Low energy effective field theory operator basis at d $\leq$ 9. *JHEP*, 06:138, 2021.

[22] Javier Fuentes-Martín, Ajdin Palavrić, and Anders Eller Thomsen. Functional matching and renormalization group equations at two-loop order. *Phys. Lett. B*, 851:138557, 2024.

[23] Adrian Carmona, Achilleas Lazopoulos, Pablo Olgoso, and Jose Santiago. Matchmakereft: automated tree-level and one-loop matching. *SciPost Phys.*, 12(6):198, 2022.

[24] Aneesh V. Manohar. Introduction to Effective Field Theories. 4 2018.

[25] Timothy Cohen. As Scales Become Separated: Lectures on Effective Field Theory. *PoS*, TASI2018:011, 2019.